Math 245 - Multivariate Calculus - Practice Examination #1

1. (20 total points) Please circle either T (true) or F (false) for each of the below statements. Each correct answer is worth 2 points. There is no partial credit or penalty for guessing. You DO NOT need to show any work. However, space is provided for any calculations you need to perform to help you decide on an answer.

I) T F The curvature κ of a straight line is 1.

II) T F For vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$, the below expressions make mathematical sense:

$$\mathbf{a} - \hat{\mathbf{a}}, \quad \mathbf{a}/|\mathbf{c}|, \quad \text{and} \qquad \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$$

III) T F The domain of $f(x, y) = \ln\left(x^2 + y^2\right)$ is $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, the set of points in $\mathbb{R}^2$ *except for* $(0, 0)$.

IV) T F For any $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$,

$$\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{u} = \mathbf{0}.$$

V) T F The equation $\mathbf{r}(t) = 1\hat{\mathbf{i}} + (2 + t)\hat{\mathbf{j}} + (3 + t^2)\hat{\mathbf{k}}$ is a line.

VI) T F A parametric representation of the ellipse $3x^2 + 4y^2 = 1$ in the plane $\mathbb{R}^2$ is

$$\mathbf{r}(t) = \frac{1}{3}(\cos t)\,\hat{\mathbf{i}} + \frac{1}{4}(\sin t)\,\hat{\mathbf{j}}, \qquad t \in [0,\, 2\pi].$$

VII) T F An equation of a hyperbolic paraboloid is $z - 2x^2 - 2y^2 = 0$.

VIII) T F $\lim_{(x,y)\to(1,-3)} (2x - 3y) = 11$.

IX) T F The distance that $\mathbf{r}(t) = (5\cos t,\, 5\sin t, 2)$ is from the origin is constant.

X) T F The area of a triangle with sides $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ is $|\mathbf{a} \times \mathbf{b}|$.

2. Let $\mathbf{a} = \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

   A) Compute $\mathbf{a} + 2\mathbf{b}$ and $\mathbf{a} \cdot \mathbf{b}$.

      Answer: $(2, 3, 5)$ *and* 3.

   B) Find the unit vector in the direction of $\mathbf{b}$.

      Answer: $(1, 1, 2)/\sqrt{6}$.

   C) Find the angle between $\mathbf{a}$ and $\mathbf{b}$.

      Answer: $\pi/6$.

3. Consider the lines

$$\mathbf{r}_1(t) = (-3, -1, 8) + (-2, -1, 3)t \quad \text{and} \quad \mathbf{r}_2(t) = (2, 5, -10) + (-5, 1, -3)t.$$

   A) Show that the two lines are orthogonal.

   B) Show that the two lines intersect. Find the point of intersection.

      Answer: The lines intersect at $(7, 4, -7)$.

4. Consider the vectors

$$\mathbf{a} = (1, -1, 2) \quad \text{and} \quad \mathbf{b} = (2, 1, 0).$$

   A) Find the equation of the plane that contains $\mathbf{a}$ and $\mathbf{b}$ and passes through the point (1,1,1).

      Answer: $-2x + 4y + 3z = 5$.

   B) Identify where the line $x = 2y = 3z$ intersects the plane from part A.

      Answer: $(x, y, z) = (5,\ 5/2,\ 5/3)$.

5. (20 points) *Carefully* graph

$$\frac{x^2}{16} + \frac{y^2}{4} + z^2 = 1$$

   in $\mathbb{R}^3$. What kind of surface is it? As part of your answer, show where (if at all), this surface intersects the x, y, and z axes.

6. Consider the parametric curve

$$\mathbf{r}(t) = t\,\hat{\mathbf{i}} + 2\cos(2\pi t)\,\hat{\mathbf{j}} + 2\sin(2\pi t)\,\hat{\mathbf{k}}.$$

Graph $\mathbf{r}(t)$ in the xyz coordinate system for $t \in [0, 2]$.

7. Find the curve of intersection between the surfaces

$$z = 3x^2 + y^2 \quad \text{and} \quad z = 16 - x^2 - 3y^2.$$

Give your answer in the form $\mathbf{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}$ for some $x(t)$, $y(t)$, and $z(t)$.

Answer: $\mathbf{r}(t) = 2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + (4 + 8\cos^2 t)\,\hat{\mathbf{k}}$.

8. Consider the curve

$$\mathbf{r}(t) = t\,\hat{\mathbf{i}} + \cosh t\,\hat{\mathbf{j}} - \sinh t\,\hat{\mathbf{k}}, \quad t \in \mathbb{R}.$$

A) Find the unit tangent vector $\hat{\mathbf{T}}(t)$ in general and at $(0, 1, 0)$.

Answer:

$$\hat{\mathbf{T}}(t) = \frac{\operatorname{sech} t\,\hat{\mathbf{i}} + \tanh t\,\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{2}} \quad \text{and} \quad \hat{\mathbf{T}}(0) = \frac{\hat{\mathbf{i}} - \hat{\mathbf{k}}}{\sqrt{2}}.$$

B) Find the length of the curve $\mathbf{r}(t)$ where $-1 \le t \le 1$.

Answer: $L = 2\sqrt{2}\sinh(1)$.

9. Find and graph the domain $\Omega \subset \mathbb{R}^2$ for the function

$$f(x, y) = \frac{1}{\ln(1 + x^2 - y^2)}.$$

Where is this function continuous? What is its range?

Solution: The domain is all points in $\mathbb{R}^2$ except when the logarithm is either undefined or zero. This happens when either

$$1 + x^2 - y^2 \le 0 \quad \text{or} \quad 1 + x^2 - y^2 = 1.$$

The set

$$\{(x, y) \in \mathbb{R}^2 \ : \ y^2 - x^2 \ge 1\}$$

is the set of points symmetric about the y-axis, on the side of the hyperbola $y^2 - x^2 = 1$ that does *not* contain the x-axis (note that $y = 0$ implies that the inequality is never satisfied by real values of x. We also want to exclude all points where $y^2 = x^2$ or on the line $y = \pm x$. Therefore, the domain is the open set

$$\boxed{\Omega = \big\{(x, y) \ : \ y^2 - x^2 < 1 \quad \text{and} \quad y \neq \pm x\big\}.}$$

Since this set is open, it follows that f is continuous on Ω. Moreover, due to the fact that there are points in Ω such that the argument of $\ln(\cdot)$ is arbitrarily small and positive/negative, as well as arbitrarily large and positive/negative, it follows that the range of $f(x, y)$ on Ω is $(-\infty, 0) \cup (0, \infty)$.

10. For the function

$$f(x, y) = \left\{ \begin{array}{ccc} \dfrac{x^2 - 2y}{y^2 + 2x} & : & (x, y) \neq (0, 0) \\ & & \\ a & : & (x, y) = (0, 0) \end{array} \right\},$$

is it possible to assign a value to $a \in \mathbb{R}$ so that the function $f(x, y)$ is continuous at (0,0)? Why or why not? Be sure to fully support your answer.

More Practice Problems

1. Consider the pair of parametric curves

$$\mathbf{r}_1(t) = (t, 1 - t, 3 + t^2) \quad \text{and} \quad \mathbf{r}_2(t) = (3 - t, t - 2, t^2).$$

A) Show that the two curves intersect. At what point in $\mathbb{R}^3$ does this occur?

Answer: At (1,0,4).

B) For $\mathbf{r}_1(t)$, find the equation of the tangent line at the point of intersection in part (A).

Answer: $x = 1 + t, \quad y = -t, \quad \text{and} \quad z = 4 + 2t.$

2. Consider the helix $\mathcal{C}$

$$\mathbf{r}(t) = 3 \sin t \,\hat{\mathbf{i}} + 3 \cos t \,\hat{\mathbf{j}} + 4t \,\hat{\mathbf{k}}$$

where $t \geq 0$.

A) Reparameterize the curve $\mathcal{C}$ in terms of arc length to deduce a formula for $\mathbf{r}(s)$ for $s \geq 0$.

Answer: $\mathbf{r}(s) = 3\,\sin\left(\frac{s}{5}\right)\,\hat{\mathbf{i}} + 3\,\cos\left(\frac{s}{5}\right)\,\hat{\mathbf{j}} + \frac{4s}{5}\,\hat{\mathbf{k}}$.

B) Explain in words the meaning of $\mathbf{r}(s)$ for a given value of s. How does it differ from $\mathbf{r}(t)$ for a given value of t?

3. Find the length of the curve

$$\mathbf{r}(t) = \ln t\,\hat{\mathbf{i}} + \frac{1}{2}t^2\hat{\mathbf{j}} + \sqrt{2}t\,\hat{\mathbf{k}}$$

when t ranges over the interval $[1, 2]$.

Answer: $L = \frac{3}{2} + \ln\,2$.

4. Consider the curve

$$\mathbf{r}(t) = t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + (1 + t^2)\,\hat{\mathbf{k}}.$$

Find the curvature $\kappa(t)$ for $\mathbf{r}(t)$ **and** find at which value of t the curvature $\kappa(t)$ is maximized.

Answer: The maximum is $\kappa = 1$ at $t = 0$.

5. Answer the following questions concerning limits in the plane:

A) Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{2xy^3}{x^2 + 8y^6}$$

does not exist.

B) Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y - x^2 - y^2}{x^2 + y^2}$$

does exist. What is its value?