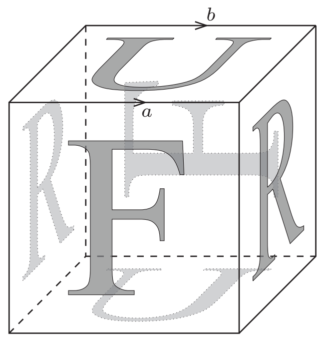
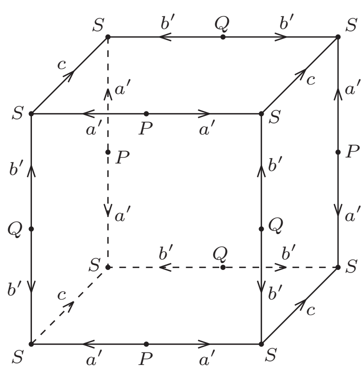
Addendum to Section 4.3 of *Topology Now!*

There is one situation in which gluing orange slices together around an edge does not result in a ball as is mentioned at the top of page 138. The following exercise illustrates this case and shows how to deal with it.

1. Consider a solid cube with front and back faces identified with a quarter twist and top and bottom faces identified with a reflection as indicated in the figure below.



   a. Show that the edge labeled a is identified with three other edges, and then is identified with itself with its orientation reversed. Show edge b suffers the same fate.
   b. Argue that the four orange-slice pieces of the neighborhood of the point P at the center of edge a are glued together to form a solid whose boundary is a projective plane rather than a sphere. Show that the same behavior occurs around the point Q at the center of edge b. It follows that this pseudo-manifold is not a manifold.
   c. Here is a way to reconcile this situation with the situation analyzed in the text where edges are glued with consistent orientations. Add two new vertices, P in the center of edge a and Q in the center of edge b. Let a' be half of a, and let b' be half of b. Check that edges and vertices are glued as shown below.



d. Check that the Euler characteristic of this pseudo-manifold is nonzero.

e. Check that the vertex S has a neighborhood bounded by the connected sum of two projective planes.

f. Argue that all interior points of edges a', b', and c have neighborhoods bounded by spheres. Hence, vertices P, Q, and S are the only points that do not have neighborhoods that are solid balls.