The No-Sided Möbius Band Robert Messer Albion College

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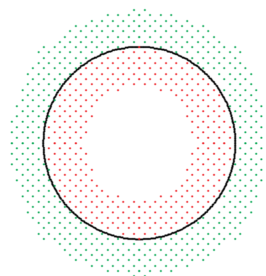
In 1946, Martin Gardner wrote a short story "The No-Sided Professor". At a meeting of the Möbius Society, an argument escalates about the existence of a no-sided Möbius band. One member folds another into a no-sided Möbius band. He passes through the fourth dimension and lands in a night club on the floor below. Pandemonium follows. Dave, have you made arrangement to cancel the Geology lab this week?

Sides of an object

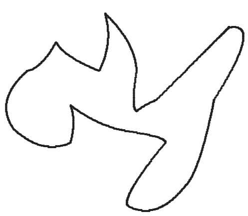
A circle S^1 in $\mathbb{R}^2$ has two sides:



The xy-plane in $\mathbb{R}^3$ has two sides (positive z-values and negative z-values).

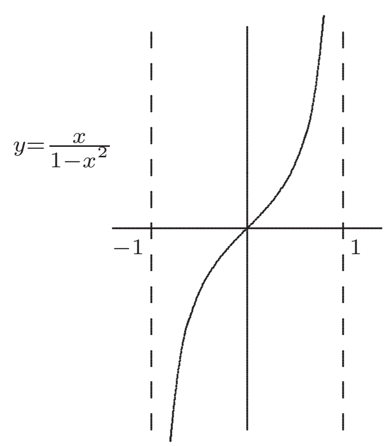
Topology

Properties of objects that are preserved under continuous deformations.



A simple-closed curve in the plane is a deformed circle.
The Jordan Curve Theorem says it has two sides.
So the number of sides of a circle in the plane is a topological property.

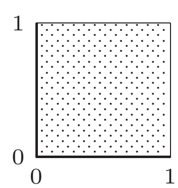
Size is not a topological property. In fact, $(-1, 1) \cong \mathbb{R}$.



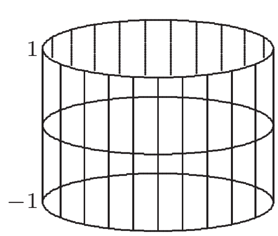
Cartesian products

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$

$[0, 1] \times [0, 1] = \{(x, y) \mid x \in [0, 1] \text{ and } y \in [0, 1]\}$



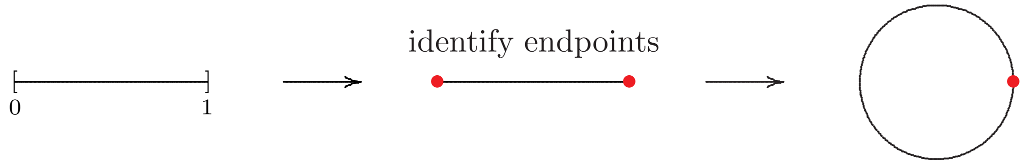
$S^1 \times [-1, 1]$

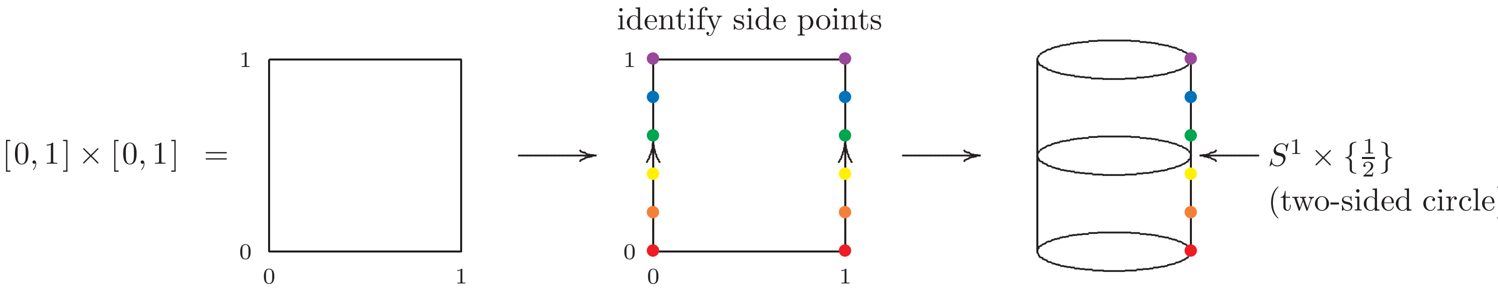


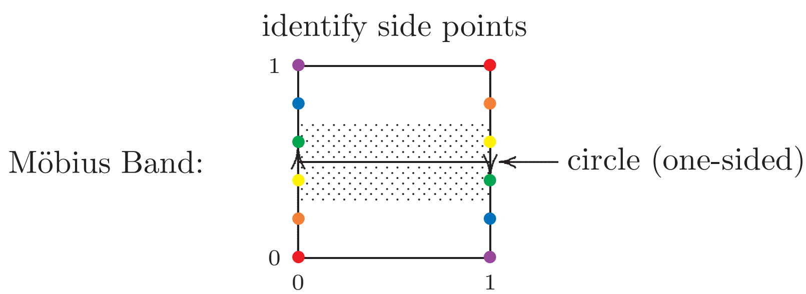
← $S^1 \times \{0\}$ is a circle with two sides (positive and negative).

Homework: What is $S^1 \times S^1$?

Pasting







We are getting a hint that number of side of an object is not an intrinsic property of the object, but rather a property of the way the object is embedded in an ambient space.

Möbius band models in $\mathbb{R}^3$

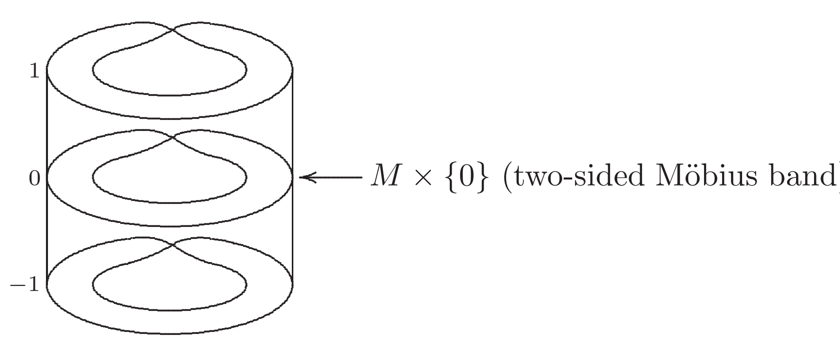
Create Möbius bands.

How many sides does your Möbius band have?

Cut down the centerline. Predict what will happen.

Two-sided Möbius band

Let M be a Möbius band. Then $M \times [-1, 1]$ has a copy of M at $M \times \{0\}$. This Möbius band has a positive side and a negative side!



A no-sided Möbius band

Let $\mathcal{L}$ be the set of lines in the plane.

Each element of $\mathcal{L}$ is a line. So $\mathcal{L}$ is a set of sets.

We want to associate each element of $\mathcal{L}$ with an ordinary point in some geometric object so we can picture all of $\mathcal{L}$.

A nonvertical line in the plane can be written $y = mx + b$.

So we can associate this line with an ordered pair (m, b).

Two problems: What about vertical lines?

Lines $y = 100x$ and $y = -100x$ are very close, but $(100, 0)$ and $(-100, 0)$ are far apart.

A nonhorizontal line in the plane can be written $x = m'y + b'$.

So we can associate this line with an ordered pair (m', b').

Example: $y = 2x + 5$ has slope 2 and y-intercept 5.

The same line $x = \frac{1}{2}y - \frac{5}{2}$ has inverse slope $\frac{1}{2}$ and x-intercept $-\frac{5}{2}$.

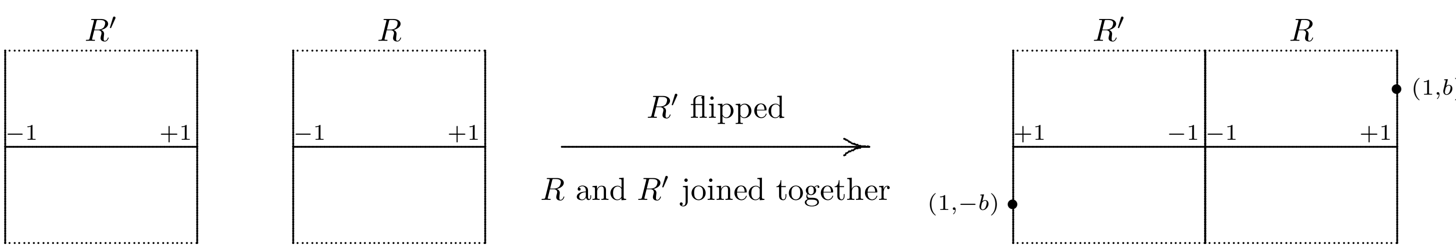
Split $\mathcal{L}$ into the set $\mathcal{A}$ of lines with slopes between -1 and 1 and the set $\mathcal{A}'$ of lines with inverse slopes between -1 and 1.

Lines in $\mathcal{A}$ correspond to points $(m, b) \in [-1, 1] \times \mathbb{R} \cong [-1, 1] \times (-1, 1) = R$.

Lines in $\mathcal{A}'$ correspond to points $(m', b') \in [-1, 1] \times \mathbb{R} \cong [-1, 1] \times (-1, 1) = R'$.

This correspondence is onto, but we have duplication of points in $R \cup R'$ for lines with slopes -1 and $+1$.

A line with slope -1, say $y = -x + b$ or $x = -y + b$, corresponds to $(-1, b)$ in both R and R'. Flip R' from left to right and join R and R' together along their $m = -1$ edges.



A line with slope 1, say $y = x + b$ or $x = y - b$, corresponds to $(1, b)$ in R and to $(1, -b)$ in R'.

So we need to paste the ends together with a half-twist.

This gives a Möbius band (with the boundary edge removed).

So $\mathcal{L}$ is topologically a Möbius band. How many sides does it have?