Mathematical Theory of Interest - Quiz #3 - Solution

1. (1 point each) Please circle either T (true) or F (false) for each of the below statements. **Answers are in BOLD.**

   A) **T** F In the amortization method of loan repayment with level payments, the amount of the principle repaid each period increases in time.

   B) **T** F For a 20-year loan of size L with monthly interest rate $j > 0$, the level monthly repayments are $L/\, a_{\overline{240}|j}$.

   C **T** F The prospective method of computing the outstanding balance on a loan requires finding the present value of all remaining loan payments.

   D) **T** F It is possible for the total interest paid on a loan to be greater than twice the loan principal.

2. (10 total points) You borrow \$4,000,000 to purchase a home in San Francisco. The nominal annual interest rate for the loan is $i^{(12)} = 6.5\%$.

   A) (5 points) If the term of the loan is 25 years, find your level monthly payments K.

   <u>Solution:</u> For $L = 4{,}000{,}000$ and $j_1 = .065/12 \simeq .0054167$, it follows that the payment is

$$K_1 = \frac{P}{a_{\overline{12\cdot 25}|j_1}} = \frac{4{,}000{,}000}{a_{\overline{300}|0.0054167}} \simeq \boxed{\$27{,}008.286.}$$

   B) (1 point) How much total interest do you pay if you complete the terms of the loan repayment over 25 years?

   <u>Solution:</u> The total interest paid is

$$25 \cdot 12 \cdot K_1 - P = \boxed{\$4{,}102{,}485.80.}$$

   C) (4 points) 15 years after the start of the loan you have the option to refinance the loan at a new interest rate of $i^{(12)} = 5.25\%$. What would be your new monthly payments?

   <u>Solution:</u> The outstanding balance after 15 years is the present value of the remaining payments under the old interest rate j_1.

$$OB_{15\cdot 12} = OB_{180} = 27008.286 \cdot a_{\overline{12\cdot 10}|j_1} \simeq \$2{,}378{,}579.228.$$

   Therefore, for $j_2 = 0.0525/12 = 0.004375$ we have new payments of

$$K_2 = \frac{OB_{180}}{a_{\overline{120}|j_2}} \simeq \boxed{\$25{,}520.18.}$$

3. (5 points) A student with school loans in the amount of \$100,000 agrees to a repayment plan that increases over time. For the first 10 years the student will repay X at the end of every month, starting one month from today. For years 11-20 the student agrees to pay $2X$ at the end of every month and for years 21-30 the student will pay $3X$ at the end of every month. Assuming an effective annual interest rate of 5.5%, find X.

   I) 215.73
   II) 217.05
   III) 219.38
   IV) 337.17
   V) 343.82

Solution: The equivalent monthly interest rate j satisfies $(1+j)^{12} = 1.055$ or $j \simeq 0.0044717$. Since the years 11-20 payments at the end of year 10 need to b discounted 10 years to $t = 0$ and the years 21-30 payments at the end of year 20 need to be discounted 20 years to $t = 0$, it follows that

$$\$100{,}000 = X \cdot a_{\overline{120}|j} + 2X \cdot a_{\overline{120}|j} \cdot \nu^{120} + 3X \cdot a_{\overline{120}|j} \cdot \nu^{240}$$

so that

$$X = \frac{\$100{,}000}{a_{\overline{120}|j} + a_{\overline{120}|j} \cdot \nu^{120} + a_{\overline{120}|j} \cdot \nu^{240}} \simeq \boxed{\$337.17.}$$

∴ **the correct answer is IV.**

4. (5 points) Hannah saves up \$5,000 as a down payment on a car that sells for P. To pay for the rest, Madison takes out a 7-year loan at $i^{(12)} = 4.9\%$ with level repayments. \$91.67 of the 31st payment pays the interest on the loan. What is P?

   I) 37,288   II) \$37,945   III) \$38,424   IV) \$38,879   V) \$39,076

Solution: The discount factor for the 7-year loan is $\nu = (1+j)^{-1} \simeq 0.995933$ with $j \simeq 0.049/12 = 0.004083$. Using the amortized premium formula with level repayment amount K, we have for the 31st payment that

$$K\left(1 - \nu^{84-31+1}\right) = \$91.67 = K\left(1 - 0.995933^{54}\right) \quad \Rightarrow \quad K \quad \Rightarrow \quad K = \frac{\$91.67}{0.995933^{52}} \simeq \$464.08.$$

It follows that the full loan amount was

$$L = \$464.08 \cdot a_{\overline{84}|j} \simeq \$32{,}943.93$$

so that the purchase price of the car is $\$32{,}943.93 + \$5{,}000 = \boxed{\$37{,}943.93.}$ **Hence, the correct answer is II.**