Math 388 - Exam 1 Practice Problems - Solutions

1. Please circle either T (true) or F (false) for each of the below statements. You do NOT need to show your work. **Answers are in BOLD.**

I) T **F** If the effective annual interest rate is 6% then $i^{(2)} = 3\%$.

II) **T** F The future value of 1 after t years is ν^{-t}.

III) **T** F If $i = 10\%$, $\ddot{s}_{\overline{10}|i} = 17.5312$.

IV) T **F** $(Ia)_{\overline{n}|i} + (Da)_{\overline{n}|i} = a_{\overline{n}|i}$.

V) T **F** $(Ia)_{\overline{\infty}|i} = \frac{1}{i^2}$.

VI) **T** F If the annual continuous interest rate is 11%, then the nominal annual rate, compounded quarterly is approximately 11.1526%

VII) T **F** If 1 is the initial value of an investment growing with force of interest $\delta(t) = t^2$, then the value of the investment at time $t = 10$ is e^{100}.

VIII) **T** F If you invest \$100 at a continuous annual interest rate is $r > 0$, in $\ln(2)/r$ years to double the account to \$200.

IX) **T** F One application of annuities is retirement planning.

X) T **F** A perpetuity due refers to the valuation of a perpetuity one period before the first payment.

2. (10 total points) You invest 4200 into an account today.

   I) If in 15 years the account is worth 7500, find the nominal annual interest rate, convertible monthly.

   Solution: The value equation for $i^{(12)}$ is

$$7500 = 4200\left(1+\frac{i^{(12)}}{12}\right)^{12\cdot 15} \quad \Rightarrow \quad \boxed{i^{(12)} \simeq 3.8717\%.}$$

   II) If, instead, the force of interest is $\delta(t) = 1/(t+2)$ and in X years your account is worth 16800, find X.

   Solution: In this case, the equation of value is

$$16800 = 4200e^{\int_0^X \frac{dt}{t+2}} = 4200e^{\ln(X+2)-\ln(2)} = 4200e^{\ln\left(\frac{X+2}{2}\right)} = 4200\cdot\frac{X+2}{2}$$

   so that

$$X+2 = \frac{16800}{2100} \quad \Rightarrow X = 8-2 = \boxed{6.}$$

3. A payment of \$X is made at time 0 that accumulates for 5 years at 7% annual effective interest. If, instead, a simple discount rate, d, is to be used, find d such that the payment of \$X at time 0 has the same accumulated value at time 5.

   A) 1.40%
   B) 3.63%
   C) 5.19%
   D) 5.74%
   E) 6.54%

   Solution: The future value of X at 7% effective annual interest is

$$X(1+0.07)^5.$$

   Using a simple discount rate d, the future value is $X(1-5d)^{-1}$. Therefore

$$X(1.07)^5 = \frac{X}{1-5d} \quad \Rightarrow \quad d = \frac{1}{5}\left(1-(1.07)^{-5}\right) \simeq 0.05740276 \simeq \boxed{5.74\%.}$$

   $\therefore$ **The correct answer is D.**

4. Consider the following:

- A one-time investment is made at time $t = 0$ into Fund A with an effective annual interest rate of 3%.
- A one-time investment is made at time $t = 0$ into Fund B with an effective annual interest rate of 2.5%.

At the end of 20 years the total amount in the two funds is \$10000. At the end of 31 years, the amount in Fund A is twice the amount in Fund B.

Find the total amount in the two funds at the end of 10 years.

A) \$5732

B) \$6602

C) \$7472

D) \$7569

E) \$8123

Solution: Let P_A and P_B be the respective amounts invested into the funds at $t = 0$. Then the equation of value of the sum of the two funds after 20 years is

$$P_A(1.03)^{20} + P_B(1.025)^{20} = 10000$$

and the relationship between the fund values after 31 years is

$$P_A(1.03)^{31} = 2P_B(1.025)^{31}.$$

Substituting for $P_A(1.03)^{20}$ in the first equation using $P_A(1.03)^{20} = 2P_B(1.025)^{31}(1.03)^{-11}$ from the second equation yields

$$2P_B(1.025)^{31}(1.03)^{-11} + P_B(1.025)^{20} = 10000$$

or

$$P_B = \frac{10000}{2(1.025)^{31}(1.03)^{-11} + (1.025)^{20}} \simeq 2107.4648 \quad \text{and} \quad P_A \simeq 3624.7344.$$

Therefore the value of the account after 10 years is

$$P_A(1.03)^{10} + P_B(1.025)^{10} \simeq \boxed{7569.073.}$$

$\therefore$ **The correct answer is D.**

5. A special type of annuity is made where \$1 is paid every three years and is paid at the beginning of each three-year period, starting at time 0. If there are n payments in this annuity, the present value can be expressed as:

A) $\dfrac{1-\nu^{3n}}{i}$ B) $\dfrac{1-\nu^{3n}}{d}$ C) $\dfrac{1-\nu^{3(n+1)}}{1-\nu}$ D) $\dfrac{1-\nu^{3n}}{1-\nu^3}$ E) $\dfrac{1-\nu^{3(n+1)}}{1-\nu^3}$

Solution: The first payment is at time $t=0$, the beginning of year 1, while the second payment is at the $t=3$, the beginning of year 4, the third payment is at $t=6$, the beginning of year 7, and so on. Therefore, the present value equation is

$$PV = 1+\nu^3+\nu^6+\nu^9+\cdot+\nu^{3(n-1)} = \sum_{j=0}^{n-1}\nu^{3j} = \sum_{j=0}^{n-1}\left(\nu^3\right)^j = \frac{1-\left(\nu^3\right)^n}{1-\nu^3} = \boxed{\frac{1-\nu^{3n}}{1-\nu^3}.}$$

∴ **The correct answer is D.**

6. Darren makes 10 annual deposits of X each into a fund earning 5% effective annual interest. The deposits accumulate to an amount that is just sufficient to allow her to withdraw \$10000 annually for 15 years, with the first withdrawal one year after the last deposit.

Find X.

Solution: The value of the account at the time of the final annual payment is $X\,s_{\overline{10}|0.05}$. Since the first withdrawal of 10000 occurs 1 year after this final deposit, the value at the time of the final deposit must match of the present value (at that time) of the 15 future annual withdrawals. That is

$$X\,s_{\overline{10}|0.05} = 10000\,a_{\overline{15}|0.05} \quad\Rightarrow\quad X = 10000\cdot\frac{a_{\overline{15}|0.05}}{s_{\overline{10}|0.05}} \simeq \boxed{8252.30.}$$

7. An annuity pays \$50 in the beginning of the first month, \$100 in the beginning of the second month, \$150 in the beginning of the third month, continuing on in this fashion for 96 months. This annuity earns a nominal annual interest rate of 12% convertible monthly. Calculate the accumulated value of the annuity one month after the final payment.

   A) \$320,000
   B) \$331,000
   C) \$339,000
   D) \$347,000
   E) \$353,000

<u>*Solution:*</u> This is an arithmetically increasing annuity with 96 payments of 50. The monthly interest rate is $i^{(12)}/12 = 0.01$. Therefore the value at the time of the last payment is

$$50\,(Is)_{\overline{96}|0.01} = 50 \cdot \frac{\ddot{s}_{\overline{96}|0.01} - 96}{0.01} \simeq 327,632.8274.$$

Therefore the value in the account one month after the final payment is

$$(1.01) \cdot (50) \cdot (Is)_{\overline{96}|0.01} \simeq (1.01)(327632.8274) \simeq \boxed{330,909.1557.}$$

$\therefore$ **The correct answer is B.**

8. Consider the two annuity payment options:

   - <u>Annuity 1</u>: Receive payments of X at the end of each year for n years. The present value of the annuity is 493 €.
   - <u>Annuity 2</u>: Receive payments of $3X$ at the end of each year for $2n$ years. The present value of the annuity is 2748 €.

   Both present values are calculated at the same annual effective interest rate. Find ν^n.

   A) 0.86
   B) 0.87
   C) 0.88
   D) 0.89
   E) 0.90

<u>*Solution:*</u> The present value equation for the first annuity is

$$493 = X\, a_{\overline{n}|i} = X \cdot \frac{1-\nu^n}{i} \quad \Rightarrow \quad \frac{X}{i} = \frac{493}{1-\nu^n}.$$

The present value equation for the second annuity is

$$2748 = 3X\, a_{\overline{2n}|i} = 3X \cdot \frac{1-\nu^{2n}}{i} \quad \Rightarrow \quad \frac{X}{i} = \frac{916}{1-\nu^{2n}}.$$

It follows that

$$\frac{493}{\cancel{(1-\nu^n)}} = \frac{916}{1-\nu^{2n}} = \frac{916}{\cancel{(1-\nu^n)}(1+\nu^n)} \quad \Rightarrow \quad \nu^n = \frac{916}{493} - 1 \simeq \boxed{0.85801217.}$$

$\therefore$ **The correct answer is A.**

9. Darius deposits \$100 at the beginning of year 1 into an account that earns 3% annual effective interest. Each subsequent payment is 5% larger than the previous payment, and he makes deposits every six months for 10 years. Find the present value of these payments at time 0.

<u>*Solution:*</u> The effective 6-month interest rate is $j = \sqrt{1.03} - 1 \simeq 0.014889$ and 20 payments are made. Therefore, the present value equation is

$$PV = 100 + 100 \cdot \frac{1.05}{1+j} + 100 \cdot \frac{1.05^2}{(1+j)^2} + \cdots + 100 \cdot \frac{1.05^{19}}{(1+j)^{19}} = 100 \sum_{k=0}^{19} \left(\frac{1.05}{1+j}\right)^k$$

or

$$PV = 100 \cdot \frac{1 - \left(\frac{1.05}{1+j}\right)^{20}}{1 - (1.05/(1+j))} \simeq \boxed{\$2816.25026}$$

10. Alexa owns a special annuity that pays her \$100 at the beginning of years 1 and 2, \$200 at the beginning of years 3 and 4, \$300 at the beginning of years 5 and 6, and so on until year 16. Calculate the accumulated value of these payments at the beginning of year 16 if the annuity pays an annual effective interest rate of 8

A) \$5,410

B) \$6,079

C) \$10,820

D) \$12,157

E) \$15,895

Solution: Increases occur every two years while two payments of equal value occur within each 2-year period. The easiest way to handle this is to first find the accumulated value of the level payments at the end of each 2-year period, and then use an increasing annuity calculation to find the accumulated value at the end of period 8, which starts at the beginning of year 14 and ends at the beginning of year 16:

- The accumulated value of two payments of 100 at the end of year two is

$$100(1.08)^2 + 100(1.08) = 224.64.$$

- The effective 2-year interest rate is $j = 1.08^2 - 1 = 0.1664$.
- The accumulated value of 8 biennial payment periods is

$$224.64\,(Is)_{\overline{8}|0.1664} = 224.64 \cdot \frac{\ddot{s}_{\overline{8}|0.1664} - 8}{0.1664} \simeq 12,156.6586.$$

∴ **The correct answer is D.**