Mathematical Theory of Interest - Examination #1 - Solution

1. (16 total points - 2 points each) Please circle either T (true) or F (false) for each of the below statements. You DO NOT have to show your work to receive credit. **If you are not sure, guess! - Answers are in BOLD.**

A) **T** F If the effective annual interest rate is $i > -1$, then the equivalent annual continuous interest rate is $r = \ln(1+i)$.

B) **T** F The present value of an increasing perpetuity immediate with monthly payments 1, 2, 3, ... and $i^{(12)} = 6\%$ is 40200.

C) **T** F $\ddot{s}_{\overline{n}|i} = (1+i)s_{\overline{n}|i}$.

D) T **F** If the annual effective discount rate is d, then the future value of \$1, n years in the future, is $\$(1+d)^{-n}$.

E) T **F** The value today of 100 payments of 1, starting one period from today, at a periodic interest rate of i, is

$$\frac{1-\nu^{101}}{1-\nu}.$$

F) T **F** Simple interest is compounded only one time per year.

G) **T** F $s_{\overline{n}|i}\nu^n = a_{\overline{n}|i}$.

H) T **F** If $i^{(4)} = 12\%$, then $i^{(2)} = 6\%$.

2. (14 total points) You invest \$2,950 into an account.

A) (7 points) In 10 years the account is worth \$5,200. What is the nominal annual interest rate convertible quarterly?

Solution: The accumulated equation of value is

$$2950\left(1+\frac{i^{(4)}}{4}\right)^{4\cdot 10} = 5200 \quad \Rightarrow \quad 1+\frac{i^{(4)}}{4} = \sqrt[40]{\frac{104}{59}}$$

or

$$i^{(2)} = 4\left(\sqrt[40]{\frac{104}{59}} - 1\right) \simeq \boxed{0.05709 \simeq 5.7\%.}$$

B) (7 points) If instead the force of interest for the account is $\delta = 0.04 + 0.0003t^2$, where t is measured in years, what is the value of the original \$2950 investment in 10 years?

Solution: The future value is

$$\begin{aligned} FV &= 2950e^{\int_0^{10}\delta(t)\,dt} = 2950e^{\int_0^{10} 0.04+0.0003t^2\,dt} = 2000e^{\left.0.04t+0.0001t^3\right|_0^{10}}, \\ &= 2950e^{0.4+0.1} = 2950e^{0.5} \simeq \boxed{\$4{,}863.73.} \end{aligned}$$

3. (14 points) You invest \$1,250 in an account earning an annual effective interest rate of r while your best friend invests the same amount into a different account that earns an annual effective discount rate of 7.5%. After 10 years your friend's account earns 20% more interest than your account. Find r.

I) 6.8%
II) 6.9%
III) 7.0%
IV) 7.1%
V) 7.2%

Solution: The future value FV_Y and interest earned I_Y by your account are

$$\text{FV}_Y = \$1250\cdot(1+r)^{10} \quad \Rightarrow \quad I_Y = \text{FV}_Y - \$1250 \simeq \$1250\cdot\left[(1+r)^{10}-1\right].$$

On the other hand, the future value of FV_{BF} and interest earned I_{BF} by your best friend's account is

$$\text{FV}_{BF} = \frac{\$1{,}250}{(1-0.075)^{10}} \simeq \$2{,}725.79 \quad \Rightarrow \quad I_{BF} = \$1{,}475.79.$$

Since $I_{BF} = 1.2 \cdot I_Y$ it follows that

$$I_{BF} = \$1{,}475.79 = 1.2 \cdot I_Y \quad \Rightarrow \quad \$1{,}475.79 = 1.2 \cdot \$1{,}250 \cdot \left[(1+r)^{10} - 1\right]$$

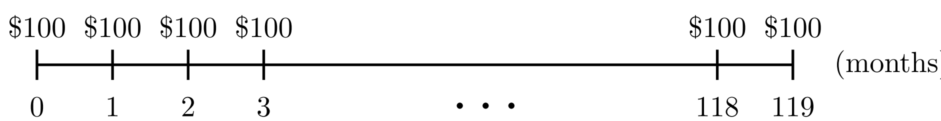
so that

$$r = \sqrt[10]{\frac{\$1{,}475.79}{1.2 \cdot \$1{,}250} + 1} - 1 \simeq 0.0709538 \simeq \boxed{7.1\%.}$$

∴ **The correct answer is IV.**

4. (14 points) A 10-year annuity-due pays 150 monthly for the first 5 years and 100 monthly for the last 5 years. The annuity earns at a nominal annual rate of 6% convertible monthly. What is the present value of this annuity?

<u>*Solution:*</u> This is the same as a 10-year annuity due with monthly payments of 100 *and* a 5-year annuity due with monthly payments of 50, each with monthly interest rate $j = 0.06/12 = 0.005 = 0.5\%$. The timeline for the first annuity is



which has present value $100\ddot{a}_{\overline{120}|0.005}$. Since the second annuity has its first payment also at time $t = 0$ so its value at time 0 is $50\ddot{a}_{\overline{60}|0.015}$. Hence, the present value of the annuity using $\nu = (1.005)^{-1}$ is

$$\begin{aligned} PV &= 100 \cdot \ddot{a}_{\overline{120}|0.005} + 50 \cdot \ddot{a}_{\overline{60}|0.005} = 50(1.005)\left[2 \cdot a_{\overline{120}|0.005} + a_{\overline{60}|0.005}\right], \\ &= 2 \cdot 50 \cdot 1.005 \cdot \frac{1-\nu^{120}}{0.005} + 50 \cdot 1.005 \cdot \frac{1-\nu^{60}}{0.005}, \\ &\simeq \boxed{\$11{,}651.59.} \end{aligned}$$

5. (14 points) Mickey receives a 10 annual payment increasing annuity paying 100 at the end of the first year and increasing by 6 each year thereafter. Billy receives a 10 annual payment decreasing annuity that pays X at the end of the first year and decreases by 7 each year thereafter. Using an annual interest rate of 6%, at time $t = 0$, Billy's annuity is worth twice as much as Mickey's annuity. Calculate X to the nearest dollar.

I) \$198

II) \$217

III) \$233

IV) \$252

V) \$276

<u>Solution:</u> Mickey's annuity can be viewed as level 10-year annuity paying 94 at the end of each year plus an increasing annuity that pays 6 the first year and an additional 6 per year for the remaining 9 years. At an annual effective rate of 6% so $\nu = (1.06)^{-1}$, this means that the present value of Mickey's annuity is

$$PV_M = \$94 \cdot a_{\overline{10}|0.06} + \$6 \cdot (Ia)_{\overline{10}|0.06} = \$94 \cdot \frac{1-\nu^{10}}{.06} + \$6 \cdot \frac{\ddot{a}_{\overline{10}|0.06} - 10\nu^{10}}{0.06} \simeq \$913.6226.$$

On the other hand, Billy's annuity can be viewed as a level annuity that pays $X + 7$ every year *minus* an increasing annuity that pays 7 per year, increasing by 7 each year thereafter. The present value of this combination is given by

$$\begin{aligned} PV_B &= (X+7)a_{\overline{10}|0.06} - 7 \cdot (Ia)_{\overline{10}|0.06}, \\ &= (X+7) \cdot (7.3601) - 7 \cdot \frac{\ddot{a}_{\overline{10}|0.06} - 10\nu^{10}}{.06}, \\ &= 7.3601 \cdot (X+7) - 258.73686, \\ &= 7.3601 \cdot X - 207.21616. \end{aligned}$$

Since $\text{PV}_B = 2 \cdot \text{PV}_M$, we have

$$7.3601 \cdot X - 207.21616 = 2 \cdot \$913.6226 \quad \Rightarrow \quad X = \frac{2 \cdot \$913.6226 + 207.21616}{7.3601} \simeq \boxed{\$276.4176.}$$

The correct answer is V.

6. (14 points) You buy an increasing perpetuity that pays \$500 at the end of the first year, \$1000 at the end of the second year, and so on. The first payment occurs 2 years from today. If the present value of this annuity is \$151,228.73, find the effective annual interest rate j.

I) 5.59%
II) 5.75%
III) 5.82%
IV) 5.86%
V) 5.91%

Solution: This is a deferred perpetuity. The value of the annuity at time $t = 1$ is $500 \cdot (1 + j)/j^2$ so that the value at time $t = 0$ is

$$\$151{,}228.73 = \frac{500 \cdot \cancel{(1+j)}}{j^2} \cdot \frac{1}{\cancel{1+j}} = \frac{500}{j^2}.$$

We conclude that

$$j = \sqrt{\frac{\$500}{\$151{,}228.73}} \simeq 0.0575 = \boxed{5.75\%.}$$

The correct answer is II.

7. (14 points) You intend to retire 40 years from January 1, 2020 with \$1,000,000 saved in an account that grows with interest $i^{(12)} = 5.5\%$. Let X be your monthly payment if you start to invest on January 31, 2020, and Y be your monthly payment if you wait 10 years and start to invest on January 31, 2030. Find $Y - X$ to the nearest dollar.

A) \$500

B) \$505

C) \$510

D) \$515

E) \$520

Solution: Starting immediately means that Y satisfies

$$X \cdot s_{\overline{40 \cdot 12}|0.055/12} = \$1{,}000{,}000 \quad \Rightarrow \quad X \simeq \$574.37,$$

while waiting 10 years requires that

$$Y \cdot s_{\overline{30 \cdot 12}|0.055/12} = \$1{,}000{,}000 \quad \Rightarrow \quad Y \simeq \$1{,}094.56.$$

Hence

$$Y - X = \$1{,}094.56 - \$574.37 \simeq \$520.19 \simeq \boxed{\$520.}$$

$\therefore$ **the correct answer is V.**