Math 245 - Quiz # 4 - Solution

Directions: Show **ALL** of your work. Answers that are not supported by calculations, graphs/diagrams, and explanations will **not** be given full credit.

1. (4 total points - 1 point each) Please circle either T (true) or F (false) for each of the below statements. There is no penalty for guessing. You DO NOT have to show your work to receive full credit. **Answers are in BOLD.**

   I) T **F** In polar coordinates, the correct limits for a circle of radius 16 in the second quadrant are $0 \leq r \leq 4$ and $\pi/2 \leq \theta \leq \pi$.

   II) T **F**

   $$\int_0^1 \int_x^2 e^x \ln(y^2+1)\, dy\, dx = \int_0^1 e^x\, dx \cdot \int_x^2 \ln(y^2+1)\, dy.$$

   III) T **F** The region $\Omega = \big\{(x, y) \ : \ 1 \leq x^2 + y^2 \leq 4\big\}$ is a Type II region.

   IV) **T** F The average value of $f(x, y) = 1$ on the unit circle is 1.

2. (6 points) Evaluate

   $$\int_0^1 \int_0^x 2e^{x^2}\, dy\, dx.$$

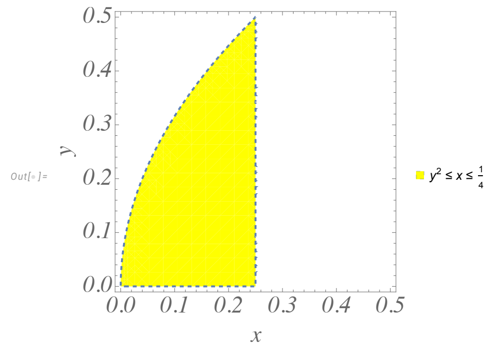
   Solution:

   $$\begin{aligned} \int_0^1 \int_0^x 2e^{x^2}\, dy\, dx &= \int_0^1 2ye^{x^2}\Big|_{y=0}^{y=x} dx = 2 \cdot \int_0^1 xe^{x^2} - 0 \cdot e^{0^2}\, dx, \\ &= 2 \cdot \underbrace{\int_0^1 xe^{x^2}\, dx}_{u=x^2,\ du=2x\, dx} = \int e^u\, du = e^{x^2}\Big|_{x=0}^{x=1} = \boxed{e-1.} \end{aligned}$$

3. (6 points) By <u>first</u> drawing the region of integration in the x-y plane, interchange the order of integration to evaluate

$$\int_0^{1/2}\int_{y^2}^{1/4} y\cos\left(16\pi x^2\right)\,dx\,dy.$$

Solution: The region Ω described by $y^2 \le x \le 1/4$ and $0 \le y \le 1/2$ has the graph shown below:



Alternatively, Ω described by equivalent inequality $0 \le y \le \sqrt{x}$ with $0 \le x \le 1/4$. The integral can thus be written as

$$\begin{aligned}\int_0^{1/2}\int_{y^2}^{1/4} y\cos\left(16\pi x^2\right)\,dx\,dy &= \int_0^{1/4}\left(\int_0^{\sqrt{x}} y\cos\left(16\pi x^2\right)\,dy\right)dx = \int_0^{1/4}\left[\frac{y^2}{2}\cdot\cos\left(16\pi x^2\right)\right]_{y=0}^{y=\sqrt{x}},\\ &= \underbrace{\frac{1}{2}\int_0^{1/4} x\cos\left(16\pi x^2\right)\,dx}_{u=16\pi x^2,\ du=32\pi x\,dx} = \frac{1}{64\pi}\int \cos u\,du = \frac{1}{64\pi}\sin\left(16\pi x^2\right)\Big|_{x=0}^{x=1/4},\\ &= \frac{1}{64\pi}\left[\sin\left(16\pi\cdot\left(\frac{1}{4}\right)^2\right) - \sin\left(16\pi\cdot 0^2\right)\right] = \frac{\sin\pi - \sin 0}{64\pi} = \boxed{0.}\end{aligned}$$

4. (5 total points) Let Ω be the region bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$ such that $y \ge 0$. Evaluate

$$\iint_\Omega \frac{dA}{1+x^2+y^2}.$$

Solution: We use polar coordinates, where the region Ω is described by the equivalent limits $1 \le r \le 2$ and $0 \le \theta \le \pi$. Then

$$\begin{aligned}\iint_\Omega \frac{dA}{1+x^2+y^2} &= \int_0^\pi\left(\int_1^2 \frac{r\,dr}{1+r^2}\right)d\theta = \int_0^\pi d\theta\cdot\underbrace{\int_1^2\frac{r\,dr}{1+r^2}}_{u=1+r^2,\ du=2r\,dr} = \pi\cdot\frac{1}{2}\int\frac{du}{u},\\ &= \frac{\pi}{2}\cdot\left[\ln\left(1+r^2\right)\right]_{r=1}^{r=2} = \frac{\pi}{2}\left(\ln 5 - \ln 2\right) = \boxed{\frac{\pi}{2}\cdot\ln\frac{5}{2}.}\end{aligned}$$