Math 245 - Quiz # 2 - Fall 2024 **Name:** ______________________

1. (4 total points - 1 point each) Please circle either T (true) or F (false) for each of the below statements. There is no penalty for guessing. You DO NOT have to show your work. **Answers are in BOLD.**

   I) **T** F $x^2 + y^2 - z^2 = 1$ describes the points on a hyperboloid of one sheet.

   II) **T** F $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}}) = \mathbf{0}$.

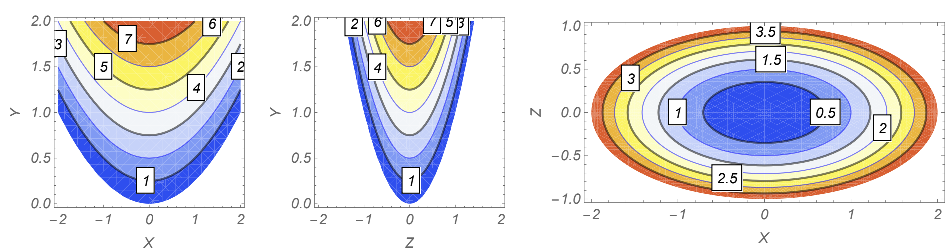
   III) **T** F The line $x = 1 + t$, $y = 1 - t$, and $z = 2 + t$ is perpendicular to the plane $x - y + z = 2$.

   IV) T **F** For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

2. (6 total points) Consider the surface $4y - x^2 = 4z^2$.

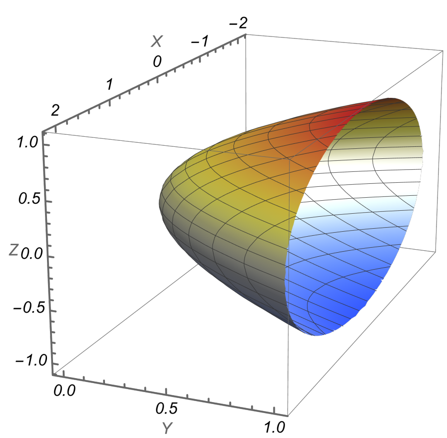
   A) (3 points) Find and sketch the xy-, xz-, and yz-traces of the surface.

   Solution: Immediately from the equation we see that the xy- and yz- traces are parabolas and the xy-traces are concentric ellipses. They are below:



   B) (3 points) Use your answers to part A) to sketch the surface in $\mathbb{R}^3$. **Identify the surface by name.**

   Solution: The surface is an elliptic paraboloid, with axis of symmetry coinciding with the y-axis.



3. (4 total points) Use the concept of the dot product to find the exact angle between $\mathbf{u} = \hat{\mathbf{i}} + \hat{\mathbf{k}}$ and $\mathbf{v} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$. Answers should be in degrees or radians.

<u>*Solution:*</u> First note that

$$|\mathbf{u}| = |\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2} \quad \text{and} \quad \mathbf{u} \cdot \mathbf{v} = (1, 0, 1) \cdot (0, 1, -1) = -1.$$

Therefore, via the unit circle, the angle θ between $\mathbf{u}$ and $\mathbf{v}$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \boxed{\frac{2\pi}{3} = 120^\circ.}$$

4. (6 points) Find the equation of the plane passing through the origin and containing the line $x = t$, $y = 1 - 3t$, $z = 2 + t$.

<u>*Solution:*</u> The the point $(0, 1, 2)$ on the line must be *in the plane*, and the direction vector of the line, $\mathbf{u} = (1, -3, 1)$, must be parallel to the plane. In addition, Since the plane also contains $\mathbf{0}$, it follows that the vector $\mathbf{v} = (0, 1, 2) - \mathbf{0} = (0, 1, 2)$ must also be parallel to the desired plane. Therefore,

$$\begin{aligned}
\mathbf{n} &= \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{vmatrix} \hat{\mathbf{i}} & -\hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -3 & 1 \\ 0 & 1 & 2 \end{vmatrix}, \\
&= \hat{\mathbf{i}}\left((-3)2 - 1 \cdot 1\right) - \hat{\mathbf{j}}\left(1 \cdot 2 - 0 \cdot 1\right) + \hat{\mathbf{k}}\left(1 \cdot 1 - 0 \cdot (-3)\right), \\
&= -7\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}.
\end{aligned}$$

The equation of the plane must then have the form $-7x - 2y + z = d$. But $d = 0$ since the plane contains the point $(0, 0, 0)$. The final answer is thus

$$\boxed{z = 7x + 2y.}$$