Math 245 - Practice Quiz # 4 - Solution

1. (4 total points - 1 point each) Please circle either T (true) or F (false) for each of the below statements. There is no penalty for guessing. You DO NOT have to show your work to receive full credit. **Answers are in BOLD.**

   I) T **F** The limits in polar coordinates for $\iint_\Omega f\,dA$ where $\Omega = \left\{(x,y)\ :\ 4 \le x^2+y^2 \le 9\right\}$ are $0 \le \theta \le 2\pi$ and $4 \le r \le 9$.

   II) **T** F If $f(x,y) = x^2 \sin\left(y^2\right)$, then

   $$\iint_{[-2,2]\times[0,\pi]} f(x,y)\,dA = \int_{-2}^{2} x^2\,dx \cdot \int_0^\pi \sin\left(y^2\right)\,dy.$$

   III) T **F** The differential area element in polar coordinates is $dA = dr\,d\theta$.

   IV) **T** F Given a smooth function $f(x,y)$ defined on $\mathbb{R}^2$, it is always true that

   $$\int_0^1 \int_0^x f(x,y)\,dy\,dx = \int_0^1 \int_y^1 f(x,y)\,dx\,dy.$$

2. Evaluate

$$\int_0^1 \int_0^{s^2} \cos\left(s^3\right)\,dt\,ds$$

Answer: $\sin(1)/3$.

Solution:

$$\begin{aligned}\int_0^1 \int_0^{s^2} \cos\left(s^3\right)\,dt\,ds &= \int_0^1 t\cdot\cos\left(s^3\right)\Big|_{t=0}^{t=s^2}\,ds = \underbrace{\int_0^1 s^2\cos\left(s^3\right)\,ds}_{u=s^3,\,du=3s^2\,ds}, \\ &= \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u = \frac{1}{3}\sin\left(s^3\right)\Big|_{s=0}^{s=1} = \boxed{\frac{1}{3}\sin(1).}\end{aligned}$$

3. Evaluate

$$\int_0^1 \int_x^1 e^{\frac{x}{y}}\, dy\, dx.$$

Answer: $(e-1)/2$.

Solution: The region is the triangle bounded by the line $y = x$, $x = 0$, and $y = 1$. It can be described as $0 \le x \le y$ and $0 \le y \le 1$. Interchanging the limits of integration we get

$$\begin{aligned} I = \int_0^1 \int_x^1 e^{\frac{x}{y}}\, dy\, dx &= \int_0^1 \int_0^y e^{\frac{x}{y}}\, dx\, dy = \int_0^1 y e^{\frac{x}{y}} \Big|_{x=0}^{x=y} dy \int_0^1 y e^1 - y e^0\, dy, \\ &= \int_0^1 (e-1) y\, dy = (e-1) \cdot \frac{y^2}{2} \Big|_{y=0}^{y=1} = \boxed{\frac{e-1}{2}.} \end{aligned}$$

4. Evaluate

$$\iint_\Omega 1 + y\, dA,$$

where Ω is the semicircular disk $\Omega := \{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 \le 9 \quad \text{and} \quad y \ge 0\}$.

Solution: The polar coordinate limits are $0 \le r \le 3$ and $0 \le \theta \le \pi$. The density function is $\rho = 1 + y = 1 + r \sin\theta$. Therefore the mass is

$$\begin{aligned} M &= \iint_\Omega \rho\, dA = \int_0^\pi \int_0^3 (1 + r\sin\theta) r\, dr\, d\theta = \int_0^\pi \left(\frac{r^2}{2} + \frac{r^3}{3} \sin\theta \right) \Big|_{r=0}^{r=3} d\theta, \\ &= \int_0^\pi \frac{9}{2} + 9\sin\theta\, d\theta = \int_0^\pi \frac{9}{2} + 9\sin\theta\, d\theta = \frac{9\pi}{2} - 9(\cos\pi - \cos(0)) = \frac{9\pi}{2} + 18, \\ &= \boxed{\frac{9(\pi+4)}{2}.}. \end{aligned}$$