Math 245 - Practice Problems for Quiz # 3 - Solution

1. Please circle either T (true) or F (false) for each of the below statements. There is no penalty for guessing. **Answers are in BOLD.**

I) T **F** The chain rule for smooth functions $u(x,y)$, $x(s,t)$, and $y(s,t)$ says that $z(s,t) = u(x(s,t), y(s,t))$ has the second partial derivative

$$\frac{\partial^2 z}{\partial t^2} = u_{xx}x_t^2 + 2u_{xy}x_t y_t + u_{yy}y_t^2.$$

II) **T** F $\lim_{(x,y)\to(2,1)}(2x - 3y) = 1$.

III) T **F** The domain of $\ln(x^2 + y^2)$ is $\mathbb{R}^2$.

IV) T **F** The chain rule for smooth functions $u(x,y)$, $x(t)$, and $y(t)$ says that $z(t) = u(x(t), y(t))$ has the derivative

$$\frac{dz}{dt} = u_{xx}x'(t) + 2u_{xy}x'(t)y'(t) + u_{yy}y'(t).$$

V) T **F** For any linear function $f(x,y)$, $f_x = f_y = 0$.

VI) T **F** If $f(x,y)$ is continuous at $(a,b) \in \mathbb{R}^2$, it must also be differentiable at (a,b).

2. Compute the below partial derivatives:

$$\frac{\partial}{\partial x}\left(x^2 y - \ln(x+y)\right) \quad \text{and} \quad \left(\sin^2\left(\frac{x+y}{x-y}\right)\right)_y.$$

Solution: Using the chain rule, we have for the first case

$$\frac{\partial}{\partial x}\left(x^2 y - \ln(x+y)\right) = 2xy - \frac{(x+y)_x}{x+y} = \boxed{2xy - \frac{1}{x+y}.}$$

For the second case find

$$\begin{aligned}
\left(\sin^2\left(\frac{x+y}{x-y}\right)\right)_y &= 2\sin\left(\frac{x+y}{x-y}\right)\cdot\cos\left(\frac{x+y}{x-y}\right)\cdot\left(\frac{x+y}{x-y}\right)_y, \\
&= \sin\left[2\left(\frac{x+y}{x-y}\right)\right]\cdot\frac{(x-y)\cdot(x+y)_y - (x+y)\cdot(x-y)_y}{(x-y)^2}, \\
&= \sin\left[2\left(\frac{x+y}{x-y}\right)\right]\cdot\frac{(x-y)\cdot 1 - (x+y)\cdot(-1)}{(x-y)^2}, \\
&= \boxed{\sin\left[2\left(\frac{x+y}{x-y}\right)\right]\cdot\frac{2x}{(x-y)^2}.}
\end{aligned}$$

3. Let $f(x,y) = \ln\left(x^2+y\right)$. Find the rate of change of f at $(1,1)$ in the direction of the vector $\mathbf{v} = -3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$.

<u>*Solution:*</u> First note that

$$\nabla f = (f_x,\, f_y) = \left(\frac{2x}{x^2+y}\right)\hat{\mathbf{i}} + \left(\frac{1}{x^2+y}\right)\hat{\mathbf{j}}$$

which, at $(x,y) = (1,1)$, has the value $\nabla f(1,1) = (1,\, 1/2)$. The unit vector in the direction of $\mathbf{v}$ is

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = (-3/5)\,\hat{\mathbf{i}} + (4/5)\,\hat{\mathbf{j}}.$$

It follows that

$$D_{\hat{\mathbf{v}}} f = \nabla f(1,1)\cdot\hat{\mathbf{v}} = (1,\, 1/2)\cdot(-3/5,\, 4/5) = -\frac{3}{5} + \frac{1}{2}\cdot\frac{4}{5} = -\frac{6}{10} + \frac{4}{10} = \boxed{-\frac{1}{5}.}$$

4. Find the equation of the plane tangent to the surface $z = e^{x+y}$ at $(0,0,1)$.

<u>*Solution:*</u> First note that

$$z_x = e^{x+y}\cdot(x+y)_x = e^{x+y} \quad \text{and} \quad z_y = e^{x+y}\cdot(x+y)_y = e^{x+y}$$

so that at $(0,0,1)$, the normal vector is

$$\mathbf{n} = (z_x,\, z_y,\, -1)\big|_{x=y=0} = \left(e^{x+y},\, e^{x+y},\, -1\right)\big|_{x=y=0} = (1,1,-1).$$

It follows that the plane is

$$1\cdot(x-0) + 1\cdot(y-0) - 1\cdot(z-1) = 0 \quad \text{or} \quad \boxed{x+y-z=-1.}$$

5. Find and graph in the x-y plane the domain of

$$f(x,y) = \frac{xy}{1-x^2-y^2}.$$

<u>*Solution:*</u> The numerator xy is well defined and makes sense on all of $\mathbb{R}^2$. In addition, the divisor is 0 when

$$x^2+y^2-1=0 \quad \Rightarrow \quad x^2+y^2=1.$$

Therefore, the only points in $\mathbb{R}^2$ that are **excluded** from the domain of f are those on the circle of radius 1 centered at $(0,0)$. Formally, we write

$$\boxed{\text{dom}(f) = \left\{(x,y)\in\mathbb{R}^2 \,:\, x^2+y^2\neq 1\right\}.}$$

The corresponding graph is the <u>interior</u> of the a circle of radius 1 centered at the origin as well as the <u>exterior</u> of a circle of radius 1 centered at the origin. That is, all of $\mathbb{R}^2$ except the unit circle.

6. Determine whether or not the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y - x^2 - y^2}{x^2+y^2}$$

exists. If it does not, prove your conclusion. If it does, demonstrate why and find its value?

Solution: The limit exists and is -1. To see this, use $x = r\cos\theta$ and $y = r\sin\theta$ recall that $(x,y)\to(0,0)$ if and only if $r\to 0$ so that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y - x^2 - y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta \cdot r\sin\theta - r^2}{r^2} = \lim_{r\to 0} r\cos^2\theta\sin\theta - 1 = \boxed{-1.}$$

7. Let $f(x,y) = x^2y + y^3x$. Compute ∇f. Use your answer to find the rate of change of $f(x,y)$ at $(1,1,2)$ in the direction $\hat{\mathbf{i}} - \hat{\mathbf{j}}$.

Solution: Since $f_x = 2xy + y^3$ and $f_y = x^2 + 3xy^2$, the gradient is

$$\nabla f = (f_x,\ f_y) = \boxed{\left(2xy+y^3\right)\hat{\mathbf{i}} + \left(x^2+3xy^2\right)\hat{\mathbf{j}}.}$$

At $(1,1,2)$,

$$\nabla f(1,1) = (2+1)\,\hat{\mathbf{i}} + (1+3)\,\hat{\mathbf{j}} = (3,4).$$

The unit vector in the direction of change is

$$\hat{\mathbf{u}} = \frac{\hat{\mathbf{i}} - \hat{\mathbf{j}}}{|\hat{\mathbf{i}} - \hat{\mathbf{j}}|} = \frac{1}{\sqrt{2}}(1,-1).$$

It follows that the direction derivative of f at $(1,1,2)$ in the direction $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ is

$$D_{\hat{\mathbf{u}}} f(1,1) = \nabla f(1,1)\cdot\hat{\mathbf{u}} = (3,4)\cdot(\frac{(1,-1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\cdot(3-4) = \boxed{-\frac{\sqrt{2}}{2}.}$$

8. Compute $\partial/\partial x$ and $\partial/\partial y$ for both $f(x,y)$ and $g(x,y)$ below:

$$f(x,y) = 3x^2y^5 - \ln\left(xy^2\right) \quad \text{and} \quad g(x,y) = \tan^{-1}\left(\frac{x}{xy+1}\right) + \tan\left(e^{x-y}\right).$$

Solution: Using the chain rule we have that

$$f_x \equiv \frac{\partial f}{\partial x} = 2\cdot 3xy^5 - \frac{\left(xy^2\right)_x}{xy^2} = 6xy^5 - \frac{y^2}{xy^2} = \boxed{6xy^5 - \frac{1}{x},}$$

and

$$f_y \equiv \frac{\partial f}{\partial y} = 3x^2 \cdot 5y^4 - \frac{(xy^2)_y}{xy^2} = 15x^2y^4 - \frac{2xy}{xy^2} = \boxed{15x^2y^4 - \frac{2}{y}.}$$

For g_x and g_y we use the chain rule and the quotient rule to get

$$\begin{aligned} g_x = \frac{\partial g}{\partial x} &= \frac{1}{\left(\frac{x}{xy+1}\right)^2 + 1} \cdot \left(\frac{x}{xy+1}\right)_x + \sec^2\left(e^{x-y}\right) \cdot \left(e^{x-y}\right)_x, \\ &= \frac{(xy+1)^2}{x^2+(xy+1)^2} \cdot \frac{(xy+1)\cdot 1 - x \cdot y}{(xy+1)^2} + e^{x-y}\sec^2\left(e^{x-y}\right), \\ &= \boxed{\frac{1}{x^2+(xy+1)^2} + e^{x-y}\sec^2\left(e^{x-y}\right),} \end{aligned}$$

and

$$\begin{aligned} g_y = \frac{\partial g}{\partial x} &= \frac{1}{\left(\frac{x}{xy+1}\right)^2 + 1} \cdot \left(\frac{x}{xy+1}\right)_y + \sec^2\left(e^{x-y}\right) \cdot \left(e^{x-y}\right)_y, \\ &= \frac{(xy+1)^2}{x^2+(xy+1)^2} \cdot \frac{-x^2}{(xy+1)^2} - e^{x-y}\sec^2\left(e^{x-y}\right), \\ &= \boxed{-\frac{x^2}{x^2+(xy+1)^2} - e^{x-y}\sec^2\left(e^{x-y}\right),} \end{aligned}$$

9. Use the chain rule to compute u_s and u_t if

$$u(x,y) = e^{x^2-y^2} \quad \text{with} \quad x(s,t) = \frac{s}{t+1} \quad \text{and} \quad y(s,t) = \sec(st).$$

<u>*Solution:*</u> The chain rules has the form

$$u_s = u_x x_s + u_y y_s \quad \text{and} \quad u_t = u_x x_t + u_y y_t.$$

Since

$$u_x = 2xe^{x^2-y^2}, \quad u_y = -2ye^{x^2-y^2},$$

with

$$x_s = \frac{1}{t+1}, \quad x_t = -\frac{s}{(t+1)^2}, \quad y_s = t\sec(s\,t)\tan(s\,t), \quad \text{and} \quad y_t = s\sec(s\,t)\tan(s\,t),$$

it follows that

$$\begin{aligned} u_s &= \left(2xe^{x^2-y^2}\right) \cdot \left(\frac{1}{t+1}\right) + \left(-2ye^{x^2-y^2}\right) \cdot \left(t\sec(s\,t)\tan(s\,t)\right), \\ &= \frac{2se^{\frac{s^2-(t+1)^2\sec^2(s\,t)}{(t+1)^2}}}{(t+1)^2} - 2t\sec^2(s\,t)\tan(s\,t)e^{\frac{s^2-(t+1)^2\sec^2(s\,t)}{(t+1)^2}}, \\ &= \boxed{e^{\frac{s^2-(t+1)^2\sec^2(s\,t)}{(t+1)^2}}\left(\frac{2s}{(t+1)^2} - 2t\sec^2(s\,t)\tan(s\,t)\right).} \end{aligned}$$

Similarly,

$$\begin{aligned}
u_t &= \left(2xe^{x^2-y^2}\right)\cdot\left(\frac{-s}{(t+1)^2}\right)+\left(-2ye^{x^2-y^2}\right)\cdot\left(s\sec(s\,t)\tan(s\,t)\right),\\
&= \frac{-2s^2e^{\frac{s^2-(t+1)^2\sec^2(s\,t)}{(t+1)^2}}}{(t+1)^3}-2s\sec^2(s\,t)\tan(s\,t)e^{\frac{s^2-(t+1)^2\sec^2(s\,t)}{(t+1)^2}},\\
&= \boxed{-2s\,e^{\frac{s^2-(t+1)^2\sec^2(s\,t)}{(t+1)^2}}\left(\frac{s}{(t+1)^3}+\sec^2(s\,t)\tan(s\,t)\right).}
\end{aligned}$$