Math 245 - Project # 3 - Exploring Surfaces with *Mathematica*

DUE: 11:59PM on Monday, November 11, 2024

Objective: To use *Mathematica* to generate surface plots and contour plots for

- functions of two Cartesian variables;
- functions described using cylindrical and spherical coordinate systems.

Questions: For all of the below problems, in addition to using Mathematical to generate images, you must provide full analytical support for your solutions.

1. Let

$$f(x, y) = \sin(\pi x y).$$

   A) Find the equation of the plane $\mathcal{P}$ that is tangent to the surface at the point $(1/2, 1/3, 1/2)$.
   B) Find the equation of the line $\mathcal{L}$ that is normal to the surface at the point $(1/2, 1/3, 1/2)$.
   C) Plot the surface $z = f(x, y)$, the plane $\mathcal{P}$, and the line $\mathcal{L}$ so that all appear in the same graph, with viewing window $[-1, 1] \times [-1, 1] \times [-1, 1]$. Plot the surface in yellow, the plane in green, and the line thick, dashed, and blue. **Note:** Be sure to choose an appropriate plotting domain for the plane so that it is small relative to the surface, and centered at the point of tangency. Also, be sure to use the "BoxRatios→ $\{1, 1, 1\}$" command to ensure that the aspect ratio of your graph is correct.

2. Consider the function family

$$f(x, y) := (x^2 + y^2)e^{(ax^2+by^2)}, \quad a, b \in \mathbb{R}.$$

   A) Use the "Plot3D" and the "ContourPlot" commands in *Mathematica* to graph the surface $z = f(x, y)$ for $a = b = 1$ over the domain $[-2, 2] \times [-2, 2]$. Use the command

   ColorFunction→ *"TemperatureMap"*

   to generate a nice color for the plot and the command "PlotPoints→ 50" to ensure that the graph is smooth.* How many maxima does the surface have? How many minima? Where are they located?

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*Make similar adjustments for the the remainder of the graphs in this project to ensure that your solution graph are both visually pleasing and accurate & smooth.

B) Identify values for a and b so that the resulting surface has exactly two distinct maxima and only one minima. Generate both a surface plot and a contour plot of the surface as you did in Part A.

C) Identify values for a and b so that the resulting surface has only one minima and no maxima. Generate both a surface plot and a contour plot of the surface as you did in Part A.

3. In addition to the Cartesian coordinate system, you can also use the *cylindrical coordinate system* to represent points in $\mathbb{R}^3$. Specifically, the Cartesian coordinates (x, y, z) are replaced by the cylindrical coordinates (r, θ, z) via

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right), \quad \text{and} \quad z = z.$$

Now, instead of representing surfaces using x and y as the independent variables and $z = f(x, y)$ as the dependent variable, we can use r and θ are the independent variables and x, y, and z are dependent variables via

$$x = x(r, \theta) = r\cos\theta, \quad y(r, \theta) = r\sin\theta, \quad \text{and} \quad z(r, \theta) = f(r, \theta).$$

Here $\theta \in [0, 2\pi)$ and $r \geq 0$.

A) Use the "ParametricPlot3D" command to plot the cylinder $r = 1$ using the functions

$$x(r, \theta) = \cos\theta, \quad y(r, \theta) = \sin\theta, \quad \text{and} \quad z(r, \theta) = z.$$

Let $z \in [-1, 1]$ so that your cylinder is 2 units in length.† Use your answer to plot another cylinder with radius 2 and axis of symmetry aligned with the x axis.

B) Plot the cone $z = \sqrt{x^2 + y^2}$ using the functions

$$x(r, \theta) = r\cos\theta, \quad y(r, \theta) = r\sin\theta, \quad \text{and} \quad z(r, \theta) = r$$

with $r \in [0, 1]$ so that the cone has a maximum radius of 1. Then use your answer to also plot a *half-cone* with maximum radius 2, axis of symmetry aligned with the y axis, $z \geq 0$, and such that it has a 'semicircular 'hole" in the bottom of the cone of radius 1/10.

C) Plot

$$f(x, y) := \sqrt{x^2 + y^2}\cos\left(\frac{1}{x^2 + y^2}\right)$$

using the functions

$$x(r, \theta) = r\cos\theta, \quad y(r, \theta) = r\sin\theta, \quad \text{and} \quad z(r, \theta) = f(x(r, \theta), y(r, \theta))$$

and the parametric domain $r \in [1/10, 1]$ and $\theta \in [0, 2\pi]$. Then answer the following questions:

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†Note that $r = 1$ is fixed in this example and z is playing the role of another parameter.

i) Describe using words, as best you can, the surface you see.
ii) What do you think is the limiting behavior of $f(x,y)$ as $(x,y) \to (0,0)$?
iii) Provide analytical proof that your conjecture from part (i) is true.

4. Another alternative to the Cartesian coordinate system is the *spherical coordinate system*, where the familiar coordinates (x,y,z) are replaced by the triple $(\rho,\ \phi,\ \theta)$, where ρ is the distance from the origin $(0,0,0)$, ϕ is the angle that the position vector to a point from the origin is from the positive z axis, and θ is the angle that the projection of the position vector from the origin to the point into the xy plane makes with the positive x axis. In particular, given (x,y,z) in Cartesian coordinates,

$$\rho = \sqrt{x^2+y^2+z^2}, \quad \phi = \cos^{-1}\left(\frac{z}{\rho}\right), \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Now, instead of using x and y as the independent variables and $z = f(x,y)$ as the dependent variable, ϕ and θ are the independent variables and x, y, and z are dependent variables. Specifically,

$$x = x(\phi,\theta) = \rho\cos\theta\sin\phi, \quad y(\phi,\theta) = \rho\sin\theta\sin\phi, \quad \text{and} \quad z(\phi,\theta) = \rho\cos\phi.$$

where generally ρ is a function of ϕ and θ, $\phi \in [0,\pi]$, and $\theta \in [0,2\pi)$.

A) Plot the sphere $\rho = 1$ using the ParametricPlot3D command and the functions

$$x = x(\phi,\theta) = \cos\theta\sin\phi, \quad y(\phi,\theta) = \sin\theta\sin\phi, \quad \text{and} \quad z(\phi,\theta) = \cos\phi.$$

and the parametric domain $\phi \in [0,\pi]$ and $\theta \in [0,2\pi]$. Use your answer to plot a hemisphere of radius 2 such that $y \le 0$.

B) Consider the family of parametric surfaces described in spherical coordinates by the system of parametric equations

$$\rho(\theta,\phi) := R + a\sin m\theta\sin n\phi$$

where $R > a > 0$ are constant real numbers, and m, n are positive integers.

i) Use the ParametricPlot3D command to plot the surface when $R = 4$, $a = 1$, $m = 2$, and $n = 1$ over the domain $\phi \in [0,\pi]$ and $\theta \in [0,2\pi]$. What happens to your graph when m is increased to 3, 4, or 5?
ii) Use the ParametricPlot3D command to plot the surface when $R = 4$, $a = 1$, $m = 1$, and $n = 2$ over the domain $\phi \in [0,\pi]$ and $\theta \in [0,2\pi]$. What happens to your graph when n is increased to 3, 4, or 5?
iii) Use the ParametricPlot3D command to plot the surface when $R = 4$, $a = 1$, $m = n = 10$ over the domain $\phi \in [0,\pi]$ and $\theta \in [0,2\pi]$. What do you think this surface resembles? Why?
iv) Pick values of a, m, n, and R that yield at least three different and "interesting" graphs. Describe what you see.