Math 245 - Multivariate Calculus - Practice Final Exam #2

1. (20 total points - 2 points each) Please circle either T (true) or F (false) for each of the below statements. You DO NOT have to show your work to receive credit. **If you are not sure, guess!**

I) T F Given $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

II) T F For a smooth vector field $\mathbf{F} = \nabla f = P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}}$, it is necessary that $Q_x = P_y$.

III) T F $f(x, y) = \sin\left(x^2 - y^2\right)$ is continuous on $\mathbb{R}^2$ except at $(0, 0)$.

IV) T F If a curve $\mathbf{r}(t)$ has $\kappa = 0$, then $\mathbf{r}' \,||\, \mathbf{r}''$.

V) T F A critical point for a function $f : \Omega \to \mathbb{R}$ is a point $\mathbf{x}$ where either $\nabla f(\mathbf{x}) = \mathbf{0}$ *or* $\nabla f(\mathbf{x})$ does not exist.

VI) T F If $\Omega \subset \mathbb{R}^2$ is the region inside the unit circle, then

$$\oint_{\partial\Omega} x\,dy = \pi.$$

VII) T F A cylinder with radius 2 and axis of symmetry aligned with the z-axis can be described by the spherical coordinate equation $\rho \cos\phi = 2$.

VIII) T F The tangent plane to $z = x^2 + y^2$ at $\mathbf{0}$ is $z = 2x + 2y$.

IX) T F The region bounded by $x = 0$, $y = 1$, and $y = x$ is an example of a region that is both Type I and Type II.

X) T F If $f(x, y) = \phi(x)\psi(y)$, and $\Omega \subset \mathbb{R}^2$ is the region bounded by $y = x^2$, $y = 0$, and $x = 1$, then

$$\iint_{\Omega} f(x, y)\,dA = \int_0^1 \int_0^{x^2} \phi(x)\psi(y)\,dy\,dx = \int_0^1 \phi(x)\,dx \cdot \int_0^{x^2} \psi(y)\,dy.$$

2. (20 total points) Consider $\mathbf{u} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $\mathbf{v} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.*

A) (7 points) Find $2\mathbf{u} + 3\mathbf{v}$.

B) (7 points) Find the unit vector in the direction of $\mathbf{u}$.

C) (6 points) Find $\mathbf{u} \cdot \mathbf{v}$.

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* *Answer:* A) $7\,\hat{\mathbf{i}} - 9\,\hat{\mathbf{j}} + 21\,\hat{\mathbf{k}}$; B) $\mathbf{u}/7$; C) 23.

3. (20 points) Describe and graph in $\mathbb{R}^3$ the surface described by the equation[†]

$$x^2 - 6x + y^2 + z^2 - 20z + 9 = 0.$$

[†] Hint: Complete the square. *Answer:* A sphere of radius 10 centered at (3, 0, 10).

4. (20 total points) Consider the curve $\mathcal{C}$ described parametrically by $\mathbf{r}(t) = 2e^t\,\hat{\mathbf{i}} + e^{2t}\,\hat{\mathbf{j}} + t\,\sin t\,\hat{\mathbf{k}}$, where $0 \leq t \leq 2$.[‡]

A) (6 points) Find

$$\int_0^2 \mathbf{r}(t)\,dt.$$

B) (7 points) Find the unit tangent vector to $\mathcal{C}$ at the point $(2, 1, 0)$.

C) (7 points) Find the equation of the plane normal to $\mathcal{C}$ at $(2, 1, 0)$.

[‡] *Answer:* A) $2\left(e^2 - 1\right)\hat{\mathbf{i}} + \frac{1}{2}\left(e^4 - 1\right)\hat{\mathbf{j}} + (\sin 2 - 2\cos 2)\,\hat{\mathbf{k}}$, B) $\hat{\mathbf{u}} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{i}} + \hat{\mathbf{j}}\right)$, C) $x + y = 3$.

5. (20 points) Evaluate the following limits if they exist. If they do not exist, demonstrate why.[§]

A) (10 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + 2y^2}$$

B) (10 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}$$

[§] *Answer:* A) DNE, B) 0.

6. (20 points) Find and classify all extrema of $f(x, y) = x^4 + 2y^2 - 4xy$ on $\mathbb{R}^2$.¶

¶ *Answer:* Local minima at $(\pm 1, \pm 1)$ and a saddle point at $(0, 0$.

7. (20 total points) Evaluate the following integrals:[‖]

A) (10 points)

$$\int_1^2 \int_0^{\ln x} xe^y \, dy \, dx$$

B) (10 points) For the below integral, $\Omega \subset \mathbb{R}^3$ is the region bounded by $x^2 + y^2 = 1$ in the first quadrant and the planes $z = 0$, $z = 2$.

$$\iiint_\Omega 3x^2yz \, dV$$

[‖] *Answer:* A) $\frac{5}{6}$, B) $\frac{2}{5}$.

8. (20 points) Denote by $\mathcal{C}$ the upper half circle $x^2 + y^2 = 25$, $y \geq 0$, with orientation counter-clockwise, starting at $(5, 0)$ and ending at $(-5, 0)$. Find**

$$\int_{\mathcal{C}} \left(x^2 - 2xy + y^2\right)\, ds.$$

** *Answer:* 125π.

9. (20 points) Let $\mathcal{C}$ denote the boundary of a square with vertices $(\pm 1, \pm 1)$ in a counterclockwise direction. Evaluate [††]

$$\oint_{\mathcal{C}} \left(x^3 + xy\right)\, dx + \left(2y^2 + 2x^2 y\right)\, dy.$$

[††] *Answer:* 0.

10. (20 total points) Let $f(x,y) = \left(x^2 - y^2\right)/2$.[‡‡]

A) (6 points) Find $\mathbf{F} = \nabla\phi$.

B) (8 points) Graph $\mathbf{F}$ in $\mathbb{R}^2$, showing enough vectors so that the "flow" of the field can be easily observed.

C) (6 points) For $\mathcal{C}$, a smooth path that starts at $A = (2,1)$ and ends at $B = (5,1)$, find

$$\oint_A^B \mathbf{F} \cdot \mathbf{dr}.$$

[‡‡] *Answer:* A) $\mathbf{F} = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$, C) $\frac{21}{2}$.