Math 245 - Multivariable Calculus - Practice Final Exam #1

Directions: Answer all questions and show **ALL** of your work. Answers that are not supported by calculations and clear explanations will **not** be given full credit.

1. Consider $\mathbf{u} = 4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\mathbf{v} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$. *

   A) Find $\mathbf{u} - 2\mathbf{v}$.

   B) Find $|\mathbf{u} - 2\mathbf{v}|$.

   C) Find $\mathbf{u} \cdot \mathbf{v}$.

   D) Find $\mathbf{u} \times \mathbf{v}$. What is $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$?

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* *Answers:* $-2\hat{\mathbf{i}} - \hat{\mathbf{k}}$, $\sqrt{5}$, 30, $2\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$.

2. Find the equation of the line that passes through (0,1,1) and is orthogonal to both $\mathbf{u} = (0, -1, 3)$ and $\mathbf{v} = (2, -1, 2)$. †

† *Answer:* $x = t, \ y = 1 + 6t, \ z = 1 + 2t.$

3. Consider the curve $\mathcal{C}$ described parametrically by

$$\mathbf{r}(t) = 4\cos t\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}} + 4\sin t\,\hat{\mathbf{k}}, \quad 0 \le t \le 4\pi.$$

A) *Carefully* graph $\mathbf{r}(t)$ in the xyz coordinate system. Indicate on your graph the locations of the points where t is an integer multiple of π.

B) Find the length of $\mathcal{C}$.[‡]

[‡] *Answer:* 20π.

4. Find the equation of the plane tangent to $z = \ln(xy)$ when $(x, y) = (e, e)$.[§]

5. Show that the function

$$u(x, y) = e^{x-2t} + \sin(x + 2t)$$

satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}.$$

[§] *Answer:* $z = (x + y)/e$.

6. Find the absolute maximum and minimums values of $f(x,y) = xy$ on the quarter-circular region $\Omega = \{(x,y) : x^2 + 3y^2 \leq 1 \quad \text{and} \quad x,\, y \geq 0\}$. Identify the locations of these maxima/minima as well as the value of f at these points. ¶

¶ *Answer:* Absolute maximum of $\sqrt{3}/6$ at $(\sqrt{2}/2,\ \sqrt{6}/6)$ and an absolute minimum of 0 on $\{0\} \times [0, \sqrt{3}/3] \cup [0,1] \times \{0\}$.

7. Evaluate the following double integrals: ‖

A)

$$\int_0^1 \int_0^x y \sin\left(\pi x^3\right) dy\, dx.$$

B)

$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy.$$

‖ *Answers:* $1/(3\pi)$ and $\left(e^8 - 1\right)/3$.

8. The semicircular disk

$$\Omega := \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 9 \quad \text{and} \quad y \geq 0\}$$

has a mass-density of $\rho(x, y) = 1 + y$ kg/m^2. Find the mass M and the center of mass $(\overline{x}, \overline{y})$ of Ω. **

** *Answer:* $M = 9(\pi + 4)/2$, $(\overline{x}, \overline{y}) = (0, (16 + 9\pi)/(16 + 4\pi))$.

9. Consider the integral

$$I := \int_{-2}^{2} \left[\int_{0}^{\sqrt{4-y^2}} \left(\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2+y^2+z^2}\, dz \right) dx \right] dy.$$

A) Carefully sketch the region of integration indicated by the limits in I.

B) Evaluate I using spherical coordinates. ††

†† *Answer:* $64\pi/9$.

10. Consider the vector field

$$\mathbf{F}(x, y) = -y\,\hat{\mathbf{i}} + x\hat{\mathbf{j}}.$$

A) Plot $\mathbf{F}(x, y)$ in $\mathbb{R}^2$.

B) Show that $\mathbf{F}$ is *not* conservative.

C) Evaluate

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathcal{C}$ is the curve $y = x^2$ between the origin and $(2, 4)$. ‡‡

‡‡ *Answer:* 8/3.

11. Show that the line integral

$$\int_{\mathcal{C}} \sin y \, dx + (x \cos y - \sin y) \, dy$$

is independent of path. Use your answer to evaluate the integral where $\mathcal{C}$ is a curve starting at $(2, 0)$ and ending at $(1, \pi)$.[§§]

[§§] *Answer:* -2.

12. Evaluate

$$\oint_{\mathcal{C}} \left(x^{3/2} - 3y\right)\, dx + \left(x - y^{2/3}\right)\, dy$$

where $\mathcal{C}$ is the positively oriented boundary of the disk $\Omega = \left\{(x, y)\ :\ x^2 + y^2 \leq 9\right\}$. ¶¶

¶¶ *Answer:* 36π.