Math 245 Multivariable Calculus - Practice Examination # 2 - Solution

1. Find and graph the domain $\Omega \subset \mathbb{R}^2$ for the function

$$f(x,y) = \frac{1}{\ln(1 + x^2 - y^2)}.$$

Where is this function continuous? What is its range?

Solution: The domain is all points in $\mathbb{R}^2$ except when the logarithm is either undefined or zero. This happens when either

$$1 + x^2 - y^2 \leq 0 \quad \text{or} \quad 1 + x^2 - y^2 = 1.$$

The set

$$\{(x,y) \in \mathbb{R}^2 \ : \ y^2 - x^2 \geq 1\}$$

is the set of points symmetric about the y-axis, on the side of the hyperbola $y^2 - x^2 = 1$ that does *not* contain the x-axis (note that $y = 0$ implies that the inequality is never satisfied by real values of x. We also want to exclude all points where $y^2 = x^2$ or on the line $y = \pm x$. Therefore, the domain is the open set

$$\boxed{\Omega = \{(x,y) \ : \ y^2 - x^2 < 1 \quad \text{and} \quad y \neq \pm x\}.}$$

Since this set is open, it follows that f is continuous on Ω. Moreover, due to the fact that there are points in Ω such that the argument of $\ln(\cdot)$ is arbitrarily small and positive/negative, as well as arbitrarily large and positive/negative, it follows that the range of $f(x,y)$ on Ω is $(-\infty,0) \cup (0,\infty)$.

2. For the function

$$f(x,y) = \left\{ \begin{array}{ccc} \dfrac{x^2 - 2y}{y^2 + 2x} & : & (x,y) \neq (0,0) \\ & & \\ a & : & (x,y) = (0,0) \end{array} \right\},$$

is it possible to assign a value to $a \in \mathbb{R}$ so that the function $f(x,y)$ is continuous at $(0,0)$? Why or why not? Be sure to fully support your answer.

Solution: The answer is **No.** To see this, note that on the line of approach $y = x$, it is clear for $x \neq 0$ that

$$f(x,y = x) = f(x,x) = \frac{x^2 - 2x}{x^2 + 2x} = \frac{x - 2}{x + 2} \to -1$$

as $x \to 0$. On the other hand, on $y = -x$ we have for $x \neq 0$ that

$$f(x,-x)=\frac{x^2+2x}{x^2+2x}=\frac{x+2}{x+2}=1$$

as $x\to 0$. Since different limits are observed as $(x,y)\to(0,0)$ on different paths, we conclude that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exists. In particular, there is **no choice of** a so that the function is continuous at (0,0).

3. Answer the following questions concerning limits in the plane:

   A) Show that the limit

$$\lim_{(x,y)\to(0,0)}\frac{2xy^3}{x^2+8y^6}$$

   does not exist.

   Solution: Certainly for $y=x$ we have

$$\lim_{(x,y)\to(0,0}\frac{2xy^3}{x^2+8y^6}=\lim_{(x,y)\to(0,0}\frac{2x^4}{x^2+8x^6}=\lim_{x\to 0}\frac{2x^2}{1+8x^6}=0.$$

   On the other hand, if $y=x^{1/3}$, then

$$\lim_{(x,y)\to(0,0}\frac{2xy^3}{x^2+8y^6}=\lim_{(x,y)\to(0,0}\frac{2x^2}{x^2+8x^2}=\lim_{x\to 0}\frac{2}{1+8}=\frac{2}{9}\neq 0.$$

   It follows that the given limit does not exist.

   B) Show that the limit

$$\lim_{(x,y)\to(0,0)}\frac{x^2y-x^2-y^2}{x^2+y^2}$$

   does exist. What is its value?

   Solution: Using $x=r\cos\theta$ and $y=r\sin\theta$ we have

$$\lim_{(x,y)\to(0,0)}\frac{x^2y-x^2-y^2}{x^2+y^2}=\lim_{r\to 0}\frac{r^2\cos^2\theta\cdot r\sin\theta-r^2}{r^2}=\lim_{r\to 0} r\cos^2\theta\sin\theta-1=\boxed{-1.}$$

4. Evaluate the following partial derivatives.

   A)

$$\frac{\partial}{\partial x}\left(\sin(xy)+x^y-\ln(x+y)\right)$$

   Solution: Using the chain rule several times we have

$$\boxed{\frac{\partial}{\partial x} \; : \; y\cos(xy) + yx^{y-1} - \frac{1}{x+y}.}$$

B)

$$\frac{\partial^2}{\partial y \partial x}\left(\sin(xy) + x^y - \ln(x+y)\right)$$

Solution: Using $\partial f/\partial x$ from part (A),

$$\frac{\partial}{\partial y}\left(y\cos(xy) + yx^{y-1} - \frac{1}{x+y}\right) = \cos(xy) - xy\sin(xy) + x^{y-1} + \frac{y}{x}x^y\ln x + \frac{1}{(x+y)^2}$$

or

$$\boxed{\cos(xy) - xy\sin(xy) + \left(\frac{y\ln x + 1}{x}\right)x^y + \frac{1}{(x+y)^2}.}$$

5. Consider the function $f(x,y) = 2\sin(2x-3y)$. Find the

(a) Find ∇f.

Solution: By definition we have

$$\nabla f = (f_x, f_y) = \boxed{(4\cos(2x-3y), \; -6\cos(2x-3y)).}$$

(b) Find the rate of change of f at $(0,\pi)$ in the direction $\hat{\mathbf{i}} + \hat{\mathbf{j}}$.

Solution: This is the direction derivative of f in the direction of $(1,1)$. In particular,

$$\nabla f(0,\pi) \cdot \left(\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{2}}\right) = (4\cos(2\cdot 0 - 3\pi), -6\cos(2\cdot 0 - 3\pi) \cdot (1,1)/\sqrt{2},$$

which becomes

$$(-4,6) \cdot (1,1)/\sqrt{2} = \boxed{\frac{2}{\sqrt{2}} = \sqrt{2}.}$$

(c) In which direction at the point $(0,\pi)$ is the rate of change of f *zero*? Give your answer as a unit vector.

Solution: The rate of change is zero when the direction $\mathbf{u}$ for the directional derivative is orthogonal to $\nabla f(0,\pi) = (-4,6)$. This is the direction either parallel or antiparallel to the direction

$$\hat{\mathbf{u}} = \frac{(6,4)}{|(6,4)|} = \frac{6\hat{\mathbf{i}} + 4\hat{\mathbf{j}}}{\sqrt{6^2+4^2}} = \frac{6\hat{\mathbf{i}} + 4\hat{\mathbf{j}}}{\sqrt{52}} = \frac{6\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}}{2\sqrt{13}} = \boxed{\frac{3\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}}{\sqrt{13}}.}$$

Note that $\mathbf{u} \cdot \nabla f(0,\pi) = 0$.

6. Find the equation of the tangent plane to the surface $z = x^2 e^{x-y}$ at (2,2,4).

<u>*Solution:*</u> Since

$$z_x = 2xe^{x-y} + x^2 e^{x-y} \quad \text{and} \quad z_y = -x^2 e^{x-y}$$

it follows that at (2,2,4)

$$z_x = 2 \cdot 2e^{2-2} + 2^2 e^{2-2} = 8 \quad \text{and} \quad z_y = -2^2 e^{2-2} = -4.$$

Therefore the normal vector to $z = f(x,y)$ at (2,2,4) is $(z_x, z_y, -1) = (8, -4, -1)$. Hence, the tangent plane is

$$8(x-2) - 4(y-2) - (z-4) = 0 \quad \Rightarrow \quad \boxed{8x - 4y - z = 4.}$$

7. Consider

$$f(x,y) = x^4 + 2y^2 - 4xy.$$

A) Find all critical points of f.

<u>*Solution:*</u> Note that

$$\nabla f = (f_x, f_y) = (4x^3 - 4y, 4y - 4x) = (0,0)$$

requires that $x = y$ and

$$4x^3 - 4x = 4x(x^2-1) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = \pm 1.$$

Therefore, there are three critical points for $f(x,y)$:

$$\boxed{\{(-1,-1), (0,0), (1,1)\}.}$$

B) Use the second derivative test to analyze your answers to part (A), identifying all maxima, minima, or saddle points.

<u>*Solution:*</u> Since

$$f_{xx} = 12x^2, \quad f_{xy} = -4, \quad \text{and} \quad f_{yy} = 4,$$

it follows that

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 12x^2 \cdot 4 - (-4)^2 = 48x^2 - 16 = 16(3x^2 - 1).$$

Therefore

(x,y)	$f_{xx} = 12x^2$	$D(x,y) = 16(3x^2 - 1)$	Classification
(-1,-1)	$12(-1)^2 = 12 > 0$	$16(3(-1)^2 - 1) = 32 > 0$	Minimum
(0,0)	$12(0)^2 = 0$	$16(3(0)^2 - 1) = -16 < 0$	Saddle Point
(1,1)	$12(1)^2 = 12 > 0$	$16(3(1)^2 - 1) = 32 > 0$	Minimum

8. Consider the function

$$f(x,y) = e^{xy}\sin(x - y).$$

Verify that $f_{xy} = f_{yx}$.

<u>*Solution:*</u> Note that

$$f_x = ye^{xy}\sin(x - y) + e^{xy}\cos(x - y) \quad \text{and} \quad f_y = xe^{xy}\sin(x - y) - e^{xy}\cos(x - y).$$

Therefore

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(ye^{xy}\sin(x - y) + e^{xy}\cos(x - y)\right), \\ &= e^{xy}\sin(x - y) + xye^{xy}\sin(x - y) - ye^{xy}\cos(x - y) + xe^{xy}\cos(x - y) + e^{xy}\sin(x - y), \\ &= \boxed{2e^{xy}\sin(x - y) + xye^{xy}\sin(x - y) + (x - y)e^{xy}\cos(x - y),} \end{aligned}$$

and

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(xe^{xy}\sin(x - y) - e^{xy}\cos(x - y)\right), \\ &= e^{xy}\sin(x - y) + xye^{xy}\sin(x - y) + xe^{xy}\cos(x - y) - ye^{xy}\cos(x - y) + e^{xy}\sin(x - y), \\ &= \boxed{2e^{xy}\sin(x - y) + xye^{xy}\sin(x - y) + (x - y)e^{xy}\cos(x - y).} \end{aligned}$$

9. Determine whether or not the function

$$u(x,y) = \sin x\, \cosh y + \cos x\, \sinh y$$

is a solution of Laplace's Equation $\Delta u = u_{xx} + u_{yy} = 0$.

<u>*Solution:*</u> Since

$$(\sin x)_{xx} = -\sin x, \quad (\cosh y)_{yy} = \cosh y, \quad (\cos x)_{xx} = -\cos x, \quad \text{and} \quad (\sinh y)_{yy} = \sinh y$$

it follows that

$$u_{xx} = -\sin x \cosh y - \cos x \sinh y \quad \text{and} \quad u_{yy} = \sin x \cosh y + \cos x \sinh y$$

so clearly

$$\boxed{\Delta u = u_{xx} + u_{yy} = -u(x,y) + u(x,y) = 0.}$$

10. The radius of a right circular cone is increasing at a rate of 7 cm/sec while its height is decreasing at a rate of 20 cm/sec.* How fast is the volume changing when $r = 45$cm and $h = 100$cm? Is the volume increasing or decreasing?

 Solution: Using the chain rule we have for $V(r,h) = \pi r^2 h/3$ that

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2\pi}{3} r h\, r' + \frac{\pi}{3} r^2\, h'.$$

 Using $r = 45$, $h = 100$, $r' = 7$, and $h' = -20$, it follows that

$$\frac{dV}{dt} = \frac{2\pi}{3}\cdot 45 \cdot 100 \cdot 7 + \frac{\pi}{3}\cdot 45^2 \cdot (-20) = \boxed{7500\,\pi\,\frac{\text{cm}^3}{\text{sec}} \simeq 23{,}561.9\,\frac{\text{cm}^3}{\text{sec}}.}$$

11. Consider

$$f(x,y) = x^3 + 3xy^2 + 3y^2 - 15x + 2.$$

 A) Find all critical points of f.
 Solution: Note that

$$\nabla f = (f_x,f_y) = (3x^2 + 3y^2 - 15, 6y(x+1)) = (0,0)$$

 requires that $y = 0$ and

$$3x^2 - 15 = 0 \quad \Rightarrow \quad x = \pm\sqrt{5},$$

 or $x = -1$ and

$$3 + 3y^2 - 15 = 0 \quad \Rightarrow \quad y^2 = 4 \quad \Rightarrow \quad y = \pm 2.$$

 Therefore, there are four critical points for $f(x,y)$:

$$\boxed{\left\{(-1,-2),(-1,2),(\sqrt{5},0),(-\sqrt{5},0)\right\}.}$$

*The volume V of a right circular cone of height $h > 0$ and radius $r > 0$ is

$$V = \frac{1}{3}\pi r^2 h.$$

B) Use the second derivative test to analyze your answers to part (A), identifying all maxima, minima, or saddle points.

Solution: Since

$$f_{xx} = 6x, \quad f_{xy} = 6y, \quad \text{and} \quad f_{yy} = 6 + 6x,$$

it follows that

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 36x(x+1) - 36y^2 = 36\left(x(x+1) - y^2\right).$$

Therefore

(x,y)	$f_{xx} = 6x$	$D(x,y) = 36(x(x+1) - y^2)$	Classification
(-1,-2)	$6(-1) = -6 < 0$	$36((-1)(-1+1) - (-2)^2) = -144 < 0$	Saddle Point
(-1,2)	$6(-1) = -6 < 0$	$36((-1)(-1+1) - (2)^2) = -144 < 0$	Saddle Point
$(\sqrt{5}$,1)	$6(\sqrt{5}) > 0$	$36(\sqrt{5}(\sqrt{5}+1) - 1^2) > 0$	Minimum
$(-\sqrt{5}$,1)	$6(-\sqrt{5}) < 0$	$36(-\sqrt{5}(-\sqrt{5}+1) - 1^2) > 0$	Maximum

12. Use Lagrange multipliers to find three positive numbers whose sum is 100 and whose product is a maximum.

Solution: The objective function is $f(x,y,z) = xyz$ and the constraint is $g(x,y,z) = x + y + z = 100$. Since

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x, y, z \geq 0 \quad \text{and} \quad x + y + z = 100\right\}$$

is a closed and bounded set in $\mathbb{R}^3$ and f is continuous, we know that a f attains its maximum on Ω. Also, since $f(x,y,z) = 0$ whenever *any* of the x, y, z are zero, and $f = 0$ is clearly *not* a maximum value of xyz, it follows that all of x,y,z are strictly positive. Now, the Lagrange multiplier condition is

$$\nabla f = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \nabla g = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} yz &= \lambda, \\ xz &= \lambda, \\ xy &= \lambda. \end{aligned}$$

It follows that

$$xyz = x\lambda = y\lambda = z\lambda \quad \Rightarrow \quad z(y - x) = y(z - x) = x(z - y) = 0.$$

Since the x, y, z are nonzero it follows that $x = y = z$, and thus the unique solution to the maximum problem is

$$x + y + z = 3x = 100 \quad \Rightarrow \quad \boxed{x = y = z = \frac{100}{3}.}$$

13. Find the rectangular box with the largest surface area in the first octant with three faces in the coordinate planes and one vertex on the plane $2x + 2y + z = 14$. Ensure that you fully justify your solution.

<u>*Solution:*</u> Since the sides of the box are parallel to the coordinate planes and each plane has two parallel faces forming the sides of the box, the objective function for the surface area is $S(x,y,z) = 2xy + 2xz + 2yz$. The constraints are that the vertex with all three coordinates *nonzero*, i.e., (x,y,z) with $x, y, z > 0$ satisifes $g(x,y,z) = 2x + 2y + z = 14$. The optimization problem is therefore

$$\max_{\substack{x,\,y,\,z>0 \\ 2x+2y+z=14}} S(x,y,z).$$

Therefore the Lagrange multiplier satisfies

$$\nabla S = \begin{pmatrix} 2y + 2z \\ 2x + 2z \\ 2x + 2y \end{pmatrix} = \lambda \nabla g = \begin{pmatrix} 2\lambda \\ 2\lambda \\ \lambda \end{pmatrix} \quad \Rightarrow \quad \begin{array}{c} 2(y+z) = 2\lambda, \\ 2(x+z) = 2\lambda, \\ 2(x+y) = \lambda. \end{array}$$

Subtracting the first two equations yields

$$2(y + z) - 2(x + z) = 2(y - x) = 2\lambda - 2\lambda = 0 \quad \Rightarrow \quad x = y.$$

It follows from the third equation that

$$2x + 2y = 2x + 2x = 4x = \lambda \quad \Rightarrow \quad x = y = \frac{\lambda}{4}.$$

Plugging $x = \lambda/4 = y$ back into either of the first two equations gives

$$2y + 2z = 2 \cdot \frac{\lambda}{4} + 2z = 2\lambda \quad \Rightarrow \quad z = \frac{1}{2} \cdot \left(2\lambda - \frac{\lambda}{2}\right) = \frac{3\lambda}{4}.$$

The constraint therefore yields

$$2x + 2y + z = \frac{2\lambda}{4} + \frac{2\lambda}{4} + \frac{3\lambda}{4} = \frac{7\lambda}{4} = 14 \quad \text{or} \quad \lambda = 8.$$

Therefore

$$\boxed{x = y = \frac{8}{4} = 2 \quad \text{and} \quad z = \frac{3 \cdot 8}{4} = 6.}$$