Math 245 Multivariable Calculus - Practice Problems for Exam # 2

1. Find and graph the domain $\Omega \subset \mathbb{R}^2$ for the function

$$f(x, y) = \frac{1}{\ln(1 + x^2 - y^2)}.$$

Where is this function continuous? What is its range?

2. For the function

$$f(x, y) = \left\{ \begin{array}{ccc} \dfrac{x^2 - 2y}{y^2 + 2x} & : & (x, y) \neq (0, 0) \\ \\ a & : & (x, y) = (0, 0) \end{array} \right\},$$

is it possible to assign a value to $a \in \mathbb{R}$ so that the function $f(x, y)$ is continuous at (0,0)? Why or why not? Be sure to fully support your answer.

Answer: No, it is not possible.

3. Answer the following questions concerning limits in the plane:

A) Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{2xy^3}{x^2 + 8y^6}$$

does not exist.

B) Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y - x^2 - y^2}{x^2 + y^2}$$

does exist. What is its value?

Answer: The limit is -1.

4. Evaluate the following partial derivatives.

A)

$$\frac{\partial}{\partial x} \left(\sin(xy) + x^y - \ln(x + y) \right)$$

Answer: $y \cos(xy) + yx^{y-1} - \frac{1}{x+y}$.

B)

$$\frac{\partial^2}{\partial y \partial x}\left(\sin(xy) + x^y - \ln(x+y)\right)$$

Answer: $\cos(xy) - xy\sin(xy) + \left(\frac{y\ln x + 1}{x}\right)x^y + \frac{1}{(x+y)^2}$

5. Consider the function $f(x, y) = 2\sin(2x - 3y)$. Find the

   (a) Find ∇f.

   Answer: $\nabla f = (4\cos(2x - 3y),\ -6\cos(2x - 3y))$.

   (b) Find the rate of change of f at $(0, \pi)$ in the direction $\hat{\mathbf{i}} + \hat{\mathbf{j}}$.

   Answer: $\sqrt{2}$.

   (c) In which direction at the point $(0, \pi)$ is the rate of change of f *zero*. Give your answer as a unit vector.

   Answer: $\frac{6\hat{\mathbf{i}}+4\hat{\mathbf{j}}}{\sqrt{50}}$.

6. Find the equation of the tangent plane to the surface $z = x^2 e^{x-y}$ at $(2, 2, 4)$.

   Answer: $8x - 4y - z = 4$.

7. Consider

$$f(x, y) = x^4 + 2y^2 - 4xy.$$

   A) Find all critical points of f.

   Answer: $(-1, -1)$, $(0, 0)$, $(1, 1)$.

   B) Use the second derivative test to analyze your answers to part (A), identifying all maxima, minima, or saddle points.

   Answer: $(-1, -1)$ is a local minimum, $(0, 0)$ is a saddle point, and $(1, 1)$ is a local minimum.

8. Consider the function

$$f(x, y) = e^{xy}\sin(x - y).$$

   Verify that $f_{xy} = f_{yx}$.

9. Determine whether or not the function

$$u(x, y) = \sin x \, \cosh y + \cos x \, \sinh y$$

is a solution of Laplace's Equation $\Delta u = u_{xx} + u_{yy} = 0$.

Answer: It does.

10. The radius of a right circular cone is increasing at a rate of 7 cm/sec while its height is decreasing at a rate of 20 cm/sec.* How fast is the volume changing when $r = 45$cm and $h = 100$cm? Is the volume increasing or decreasing?

Answer: $\frac{dV}{dt} = 7500\,\pi\,\frac{\text{cm}^3}{\text{sec}} \simeq 23,561.9\,\frac{\text{cm}^3}{\text{sec}}$.

11. Consider

$$f(x, y) = x^3 + 3xy^2 + 3y^2 - 15x + 2.$$

A) Find all critical points of f.

Answer: $(-1, -2)$, $(-1, 2)$, $(\sqrt{5}, 0)$, $(-\sqrt{5}, 0)$.

B) Use the second derivative test to analyze your answers to part (A), identifying all maxima, minima, or saddle points.

Answer: $(-1, \pm 2)$ are saddle points, $(\sqrt{5}, 0)$ is a minimum and $(-\sqrt{5}, 0)$ is a maximum.

12. Use Lagrange multipliers to find three positive numbers whose sum is 100 and whose product is a maximum.

Answer: $x = y = z = 100/3$. The maximum product is $10^6/27$.

13. Find the rectangular box with the largest surface area in the first octant with three faces in the coordinate planes and one vertex on the plane $2x + 2y + z = 14$. Ensure that you fully justify your solution.

Answer: Maximum surface are of 56 with $x = 2$, $y = 2$, and $z = 6$.

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*The volume V of a right circular cone of height $h > 0$ and radius $r > 0$ is

$$V = \frac{1}{3}\pi r^2 h.$$