Math 245 - Multivariate Calculus - Practice Exam #1 - Solution

1. Please circle either T (true) or F (false) for each of the below statements. Each correct answer is worth 2 points. There is no partial credit or penalty for guessing. You DO NOT need to show any work. However, space is provided for any calculations you need to perform to help you decide on an answer. **Answers are in BOLD.**

I) T **F** The curvature κ of a straight line is 1.

II) **T** F For vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$, the below expressions make mathematical sense:

$$\mathbf{a} - \hat{\mathbf{a}}, \quad \mathbf{a}/|\mathbf{c}|, \quad \text{and} \quad \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$$

III) **T** F The minimum value of $s(t) = \int_1^t |\mathbf{r}'(t)|\, dt$ for $t \geq 1$ is 0.

IV) **T** F For any $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$,

$$\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{u} = \mathbf{0}.$$

V) T **F** The equation $\mathbf{r}(t) = 1\hat{\mathbf{i}} + (2+t)\hat{\mathbf{j}} + (3+t^2)\hat{\mathbf{k}}$ is a line.

VI) T **F** A parametric representation of the ellipse $3x^2 + 4y^2 = 1$ in the plane $\mathbb{R}^2$ is

$$\mathbf{r}(t) = \frac{1}{3}(\cos\, t)\,\hat{\mathbf{i}} + \frac{1}{4}(\sin\, t)\,\hat{\mathbf{j}}, \qquad t \in [0,\, 2\pi].$$

VII) T **F** An equation of a hyperbolic paraboloid is $z - 2x^2 - 2y^2 = 0$.

VIII) T **F** The vector $\mathbf{u} = (1, 1, 1)$ is a unit vector.

IX) **T** F The distance that $\mathbf{r}(t) = (5\cos\, t,\, 5\sin\, t, 2)$ is from the origin is constant.

X) T **F** The area of a triangle with sides $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ is $|\mathbf{a} \times \mathbf{b}|$.

2. Let $\mathbf{a} = \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

A) Compute $\mathbf{a} + 2\mathbf{b}$ and $\mathbf{a} \cdot \mathbf{b}$.

Solution:

$$\mathbf{a} + 2\mathbf{b} = (0,1,1) + 2(1,1,2) = \boxed{(2,3,5)}$$

and

$$\mathbf{a} \cdot \mathbf{b} = (0,1,1) \cdot (1,1,2) = 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 = \boxed{3.}$$

B) Find the unit vector in the direction of $\mathbf{b}$.

Solution: Since

$$|\mathbf{b}| = \sqrt{1^2 + 1^2 + 2^2} = 6,$$

it follows that

$$\hat{\mathbf{b}} = \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{(1,1,2)}{\sqrt{6}} = \boxed{\frac{1}{\sqrt{6}}\left(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}\right).}.$$

C) Find the angle between $\mathbf{a}$ and $\mathbf{b}$.

Solution: Using the definition $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ we find

$$\theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) = \cos^{-1}\left(\frac{3}{\sqrt{2}\sqrt{6}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi}{6}.}$$

3. Consider the lines

$$\mathbf{r}_1(t) = (-3,-1,8) + (-2,-1,3)t \quad \text{and} \quad \mathbf{r}_2(t) = (2,5,-10) + (-5,1,-3)t.$$

A) Show that the two lines are orthogonal.

Solution: Note that the direction vectors $\mathbf{u}_1 = (-2,-1,3)$ and $\mathbf{u}_2 = (-5,1,-3)$ satisfy

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = (-2,-1,3)\cdot(-5,1,-3) = (-2)\cdot(-5)+(-1)\cdot(1)+(3)\cdot(-3) = 10-1-9 = 0.$$

It follows that $\mathbf{u}_1 \perp \mathbf{u}_2$.

B) Show that the two lines intersect. Find the point of intersection.

Solution: Intersection requires that there are values s and t such that $\mathbf{r}_1(t) = \mathbf{r}_2(s)$, or by component,

$$-3-2t = 2-5s, \qquad -1-t = 5+s, \quad \text{and} \quad 8+3t = -10-3s.$$

Solving for s in the second equation gives $s = -6-t$ so that the first equation is

$$-3-2t = 2-5s = 2-5(-6-t) = 2+30+5t \quad \Rightarrow \quad -35 = 7t \quad \Rightarrow \quad t = -5$$

and $s = -6-t = -6-(-5) = -1$. Intersection then requires that $(s,t) = (-1,-5)$ satisfies the third equation. To check this note that

$$8+3t = 8+3(-5) = 8-15 = -7 \quad \text{and} \quad -10-3s = -10-3(-1) = -10+3 = -7.$$

Therefore the point of intersection is

$$\begin{aligned} (x,y,z) &= (-3-2t,-1-t,8+3t)\,|_{t=-5}, \\ &= (-3-2(-5),-1-(-5),8+3(-5)), \\ &= \boxed{(7,4,-7).} \end{aligned}$$

4. Consider the vectors

$$\mathbf{a} = (1, -1, 2) \quad \text{and} \quad \mathbf{b} = (2, 1, 0).$$

A) Find the equation of the plane that contains **a** and **b** and passes through the point (1,1,1).

<u>*Solution:*</u> For the plane t contain **a** and **b** its normal must be perpendicular to both. Therefore, a normal vector is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & -\hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = (0-2)\,\hat{\mathbf{i}} - (0-4)\,\hat{\mathbf{j}} + (1+2)\,\hat{\mathbf{k}} = -2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}.$$

The plane is therefore

$$-2(x-1) + 4(y-1) + 3(z-1) = 0 \quad \Rightarrow \quad -2x + 4y + 3z = 4 + 3 - 2$$

or

$$\boxed{-2x + 4y + 3z = 5.}$$

B) Identify where the line $x = 2y = 3z$ intersects the plane from part A.

<u>*Solution:*</u> The line $x = 2y = 3z$ has the parametric form $x = t$, $y = t/2$, and $z = t/3$. Therefore it intersects the plane when

$$-2t + 4(t/2) + 3(t/3) = 5 \quad \Rightarrow \quad t = 5.$$

The point of intersection is therefore

$$\boxed{(x, y, z) = (5,\ 5/2,\ 5/3).}$$

5. *Carefully* graph

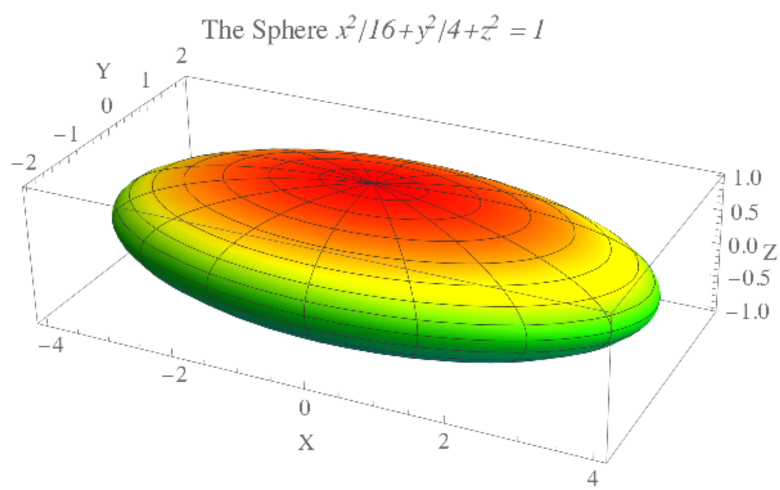
$$\frac{x^2}{16} + \frac{y^2}{4} + z^2 = 1$$

in $\mathbb{R}^3$. What kind of surface is it? As part of your answer, show where (if at all), this surface intersects the x, y, and z axes.

<u>*Solution:*</u> The surface is an ellipsoid. The intersections with the coordinate axes are achieved by setting two of the three variables equal to 0 and solving for the remaining variable:

$$\begin{aligned} x = y = 0 \quad &\Rightarrow \quad z^2 = 1 \quad \text{or} \quad z = \pm 1, \\ x = z = 0 \quad &\Rightarrow \quad y^2/4 = 1 \quad \text{or} \quad y = \pm 2, \\ y = z = 0 \quad &\Rightarrow \quad x^2/16 = 1 \quad \text{or} \quad x = \pm 4. \end{aligned}$$

The graph is

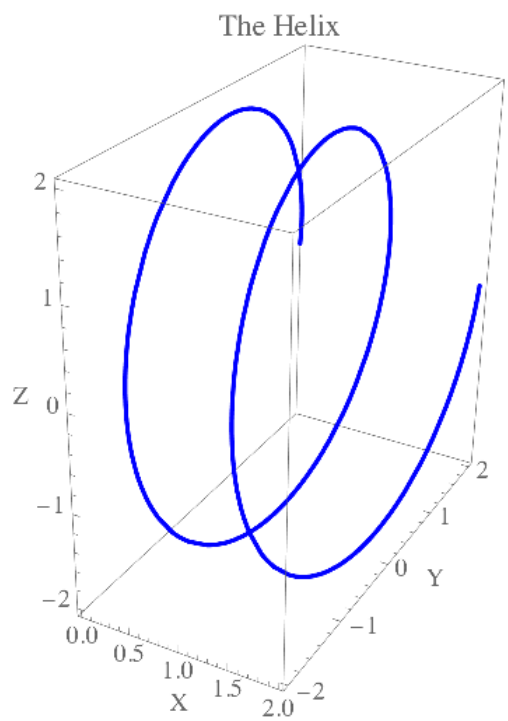


6. Consider the parametric curve

$$\mathbf{r}(t) = t\,\hat{\mathbf{i}} + 2\cos(2\pi t)\,\hat{\mathbf{j}} + 2\sin(2\pi t)\,\hat{\mathbf{k}}.$$

Graph $\mathbf{r}(t)$ in the xyz coordinate system for $t \in [0, 2]$.

<u>*Solution:*</u> This is a helix with axis of symmetry corresponding to the x axis.



Note that <u>two</u> revolutions around the x-axis occurs as t increases from $t = 0$ to $t = 2$:

7. (20 points) Find the curve of intersection between the surfaces

$$z = 3x^2 + y^2 \quad \text{and} \quad z = 16 - x^2 - 3y^2.$$

Give your answer in the form $\mathbf{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}$ for some $x(t)$, $y(t)$, and $z(t)$.

Solution: Equality of the surfaces requires that

$$z = 3x^2 + y^2 = 16 - x^2 - 3y^2 \quad \Rightarrow \quad 4x^2 + 4y^2 = 16 \quad \Rightarrow \quad x^2 + y^2 = 4.$$

This is a circle of radius of 2 that has the parameterization of

$$x(t) = 2\cos t \quad y(t) = 2\sin t.$$

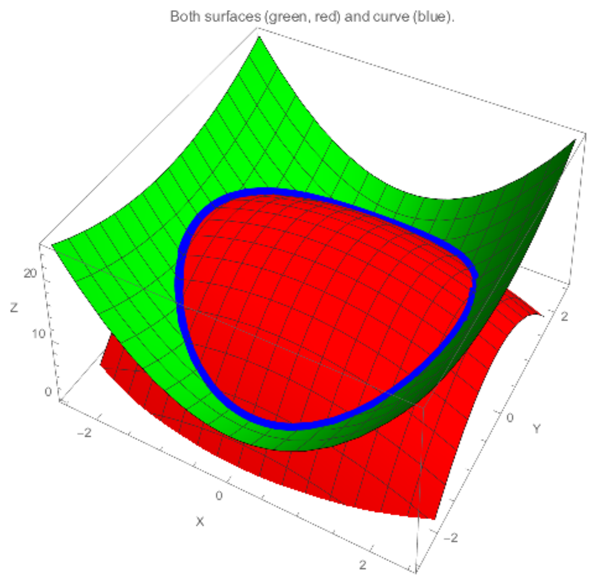
It follows that

$$z(t) = 3x^2(t) + y^2(t) = 3(2\cos t)^2 + (2\sin t)^2 = 12\cos^2 t + 4\sin^2 t = 4 + 8\cos^2 t.$$

The curve of intersection is therefore

$$\boxed{\mathbf{r}(t) = 2\,\cos\,t\,\hat{\mathbf{i}} + 2\,\sin\,t\,\hat{\mathbf{j}} + \left(4 + 8\cos^2 t\right)\,\hat{\mathbf{k}}.}$$

Although not asked for, the graph of this curve on the two surfaces is below for your enjoyment:



8. Consider the curve

$$\mathbf{r}(t) = t\,\hat{\mathbf{i}} + \cosh\, t\,\hat{\mathbf{j}} - \sinh\, t\,\hat{\mathbf{k}}, \quad t \in \mathbb{R}.$$

A) Find the unit tangent vector $\hat{\mathbf{T}}(t)$ in general and at $(0, 1, 0)$.

Solution: Since $\mathbf{r}'(t) = \hat{\mathbf{i}} + \sinh\, t\,\hat{\mathbf{j}} - \cosh\, t\,\hat{\mathbf{k}}$, $\cosh\, t > 0$ for $t \in \mathbb{R}$, and $\cosh^2 t - \sinh^2 t = 1$, we have

$$|\mathbf{r}'(t)| = \sqrt{1 + \sinh^2 t + \cosh^2 t} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh\, t$$

so that the unit tangent vector is

$$\hat{\mathbf{T}} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{(1, \sinh\, t, -\cosh\, t)}{\sqrt{2}\cosh\, t} = \boxed{\frac{1}{\sqrt{2}}\left[\operatorname{sech} t\,\hat{\mathbf{i}} + \tanh\, t\,\hat{\mathbf{j}} - \hat{\mathbf{k}}\right]] \quad \text{and} \quad \hat{\mathbf{T}}(0) = \frac{\hat{\mathbf{i}} - \hat{\mathbf{k}}}{\sqrt{2}}.}$$

B) Find the length of the curve $\mathbf{r}(t)$ where $-1 \le t \le 1$.

Solution: Using part (A), the length of the curve over the given interval is

$$\begin{aligned} L &= \int_{-1}^{1} |\mathbf{r}'(t)|\, dt = \int_{-1}^{1} \sqrt{2}\, \cosh\, t\, dt = \sqrt{2}\, \sinh\, t\,\Big|_{t=-1}^{t=1}, \\ &= \sqrt{2}\,[\sinh(1) - \sinh(-1)] = \sqrt{2}(\sinh(1) - (-\sinh(1))), \\ &= \boxed{2\sqrt{2}\sinh(1).} \end{aligned}$$

More Practice Problems

1. Consider the pair of parametric curves

$$\mathbf{r}_1(t) = (t, 1-t, 3+t^2) \quad \text{and} \quad \mathbf{r}_2(t) = (3-t, t-2, t^2).$$

A) Show that the two curves intersect. At what point in $\mathbb{R}^3$ does this occur?

<u>*Solution:*</u> If the two curves intersect then there exists $s,\, t \in \mathbb{R}$ such that $r_1(s) = r_2(t)$. Thus, for some $s,\, t \in \mathbb{R}$,

$$s = 3-t, \quad 1-s = t-2, \quad \text{and} \quad 3+s^2 = t^2.$$

Plugging s from the first equation into s in the second equation yields

$$1-(3-t) = t-2 \quad \text{or} \quad -2 = -2 \text{ (tautology)}.$$

The third equation becomes

$$3+s^2\big|_{s=3-t} = 3+(3-t)^2 = 3+9-6t+t^2 = t^2 \quad \Rightarrow \quad 12 = 6t \quad \text{or} \quad t = 2.$$

Therefore $s = 3-t = 3-2 = 1$. It follows that the point of intersection is therefore

$$\boxed{\mathbf{r}_1(1) = (1, 1-1, 3+1^2 = (1,0,4) = (3-2,\ 2-2,\ 2^2) = \mathbf{r}_2(2).}$$

B) For $\mathbf{r}_1(t)$, find the equation of the tangent line at the point of intersection in part (A).

<u>*Solution:*</u> The tangent vector to the curve $r_1(t)$ at the point of intersection is

$$r_1'(1) = (1,-1,2t)\big|_{t=1} = (1,-1,2).$$

It follows that the tangent line has the form

$$r_{\tan}(t) = (1,0,4)+(1,-1,2)t \quad \text{or} \quad \boxed{x = 1+t, \quad y = -t, \quad \text{and} \quad z = 4+2t.}$$

2. Consider the helix $\mathcal{C}$

$$\mathbf{r}(t) = 3\sin t\,\hat{\mathbf{i}} + 3\cos t\,\hat{\mathbf{j}} + 4t\,\hat{\mathbf{k}}$$

where $t \geq 0$.

A) Reparameterize the curve $\mathcal{C}$ in terms of arc length to deduce a formula for $\mathbf{r}(s)$ for $s \geq 0$.

Solution: Arc length $s(t)$ in terms of t for this curve is computed by the formula

$$s(t) = \int_0^t |\mathbf{r}'(\alpha)|\, d\alpha,$$

where α is a dummy variable of integration. Since

$$\mathbf{r}'(\alpha) = 3\,\cos\alpha\,\hat{\mathbf{i}} - 3\,\sin\alpha\,\hat{\mathbf{j}} + 4\hat{\mathbf{k}} \quad \Rightarrow \quad |\mathbf{r}'(\alpha)| = \sqrt{9\cos^2\alpha + 9\sin^2\alpha + 16} = \sqrt{9+16} = 5,$$

we conclude

$$s(t) = \int_0^t 5\, d\alpha = 5(t-0) = 5t \quad \Rightarrow \quad t = \frac{s}{5}.$$

It follows that $\mathbf{r}(t)$ reparameterized by arc length has the form

$$\boxed{\mathbf{r}(s) = 3\,\sin\left(\frac{s}{5}\right)\,\hat{\mathbf{i}} + 3\,\cos\left(\frac{s}{5}\right)\,\hat{\mathbf{j}} + \frac{4s}{5}\,\hat{\mathbf{k}}.}$$

B) Explain in words the meaning of $\mathbf{r}(s)$ for a given value of s. How does it differ from $\mathbf{r}(t)$ for a given value of t?

Solution: $\mathbf{r}(s)$ is the point in space through which the curve passes after a distance s has been traced along the helix. This differs from $\mathbf{r}(t)$ as this is the point in space through which the helix passes after t units of the parameter has passed – but t has no intrinsic physical meaning unlike s which has units of distance along the curve from $\mathbf{r}(0)$.

3. Find the length of the curve

$$\mathbf{r}(t) = \ln t\,\hat{\mathbf{i}} + \frac{1}{2}t^2\hat{\mathbf{j}} + \sqrt{2}t\,\hat{\mathbf{k}}$$

when t ranges over the interval $[1, 2]$.

Solution: Since

$$\mathbf{r}'(t) = \frac{1}{t}\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}},$$

the length L of the curve is

$$L = \int_1^2 \sqrt{\frac{1}{t^2} + t^2 + 2}\, dt = \int_1^2 \sqrt{\left(t + \frac{1}{t}\right)^2}\, dt = \int_1^2 t + \frac{1}{t}\, dt = \left(\frac{t^2}{2} + \ln\, t\right)\Bigg|_{t=1}^{t=2},$$

or

$$L = \frac{2^2}{2} + \ln 2 - \frac{1^2}{2} - \ln 1 = \boxed{\frac{3}{2} + \ln 2.}$$

4. Consider the curve

$$\mathbf{r}(t) = t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + (1+t^2)\,\hat{\mathbf{k}}.$$

Find the curvature $\kappa(t)$ for $\mathbf{r}(t)$ **and** find at which value of t the curvature $\kappa(t)$ is maximized.

Solution: First note that

$$\mathbf{r}'(t) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}, \quad \mathbf{r}''(t) = 2\,\hat{\mathbf{k}}, \quad \text{and} \quad \mathbf{r}' \times \mathbf{r}'' = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}}.$$

It follows that the curvature κ is

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{2\sqrt{2}}{(2+4t^2)^{\frac{3}{2}}} = \frac{1}{(1+2t^2)^{\frac{3}{2}}}.$$

Since the numerator of κ is constant and the denominator is minimized when $t = 0$, it follows that the curvature of $\mathbf{r}(t)$ is maximized at $\boxed{\mathbf{r}(0) = (0,0,1).}$