Math 245 - Multivariate Calculus - Practice Examination #1

1. (20 total points) Please circle either T (true) or F (false) for each of the below statements. Each correct answer is worth 2 points. There is no partial credit or penalty for guessing. You DO NOT need to show any work. However, space is provided for any calculations you need to perform to help you decide on an answer.

I) T F The curvature κ of a straight line is 1.

II) T F For vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$, the below expressions make mathematical sense:

$$\mathbf{a} - \hat{\mathbf{a}}, \quad \mathbf{a}/|\mathbf{c}|, \quad \text{and} \qquad \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$$

III) T F The minimum value of $s(t) = \int_1^t |\mathbf{r}'(t)|\, dt$ for $t \geq$ is 0.

IV) T F For any $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$,

$$\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{u} = \mathbf{0}.$$

V) T F The equation $\mathbf{r}(t) = 1\hat{\mathbf{i}} + (2+t)\hat{\mathbf{j}} + (3+t^2)\hat{\mathbf{k}}$ is a line.

VI) T F A parametric representation of the ellipse $3x^2 + 4y^2 = 1$ in the plane $\mathbb{R}^2$ is

$$\mathbf{r}(t) = \frac{1}{3}(\cos t)\,\hat{\mathbf{i}} + \frac{1}{4}(\sin t)\,\hat{\mathbf{j}}, \qquad t \in [0,\, 2\pi].$$

VII) T F An equation of a hyperbolic paraboloid is $z - 2x^2 - 2y^2 = 0$.

VIII) T F The vector $\mathbf{u} = (1, 1, 1)$ is a unit vector.

IX) T F The distance that $\mathbf{r}(t) = (5\cos t,\, 5\sin t, 2)$ is from the origin is constant.

X) T F The area of a triangle with sides $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ is $|\mathbf{a} \times \mathbf{b}|$.

2. Let $\mathbf{a} = \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

   A) Compute $\mathbf{a} + 2\mathbf{b}$ and $\mathbf{a} \cdot \mathbf{b}$.

      <u>Answer:</u> $(2, 3, 5)$ *and* 3.

   B) Find the unit vector in the direction of $\mathbf{b}$.

      <u>Answer:</u> $(1, 1, 2)/\sqrt{6}$.

   C) Find the angle between $\mathbf{a}$ and $\mathbf{b}$.

      <u>Answer:</u> $\pi/6$.

3. Consider the lines

$$\mathbf{r}_1(t) = (-3, -1, 8) + (-2, -1, 3)t \quad \text{and} \quad \mathbf{r}_2(t) = (2, 5, -10) + (-5, 1, -3)t.$$

   A) Show that the two lines are orthogonal.

   B) Show that the two lines intersect. Find the point of intersection.

      <u>Answer:</u> The lines intersect at $(7, 4, -7)$.

4. Consider the vectors

$$\mathbf{a} = (1, -1, 2) \quad \text{and} \quad \mathbf{b} = (2, 1, 0).$$

   A) Find the equation of the plane that contains $\mathbf{a}$ and $\mathbf{b}$ and passes through the point (1,1,1).

      <u>Answer:</u> $-2x + 4y + 3z = 5$.

   B) Identify where the line $x = 2y = 3z$ intersects the plane from part A.

      <u>Answer:</u> $(x, y, z) = (5,\ 5/2,\ 5/3)$.

5. (20 points) *Carefully* graph

$$\frac{x^2}{16} + \frac{y^2}{4} + z^2 = 1$$

   in $\mathbb{R}^3$. What kind of surface is it? As part of your answer, show where (if at all), this surface intersects the x, y, and z axes.

6. Consider the parametric curve

$$\mathbf{r}(t) = t\,\hat{\mathbf{i}} + 2\cos(2\pi t)\,\hat{\mathbf{j}} + 2\sin(2\pi t)\,\hat{\mathbf{k}}.$$

Graph $\mathbf{r}(t)$ in the xyz coordinate system for $t \in [0, 2]$.

7. Find the curve of intersection between the surfaces

$$z = 3x^2 + y^2 \quad \text{and} \quad z = 16 - x^2 - 3y^2.$$

Give your answer in the form $\mathbf{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}$ for some $x(t)$, $y(t)$, and $z(t)$.

Answer: $\mathbf{r}(t) = 2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + (4 + 8\cos^2 t)\,\hat{\mathbf{k}}$.

8. Consider the curve

$$\mathbf{r}(t) = t\,\hat{\mathbf{i}} + \cosh t\,\hat{\mathbf{j}} - \sinh t\,\hat{\mathbf{k}}, \quad t \in \mathbb{R}.$$

A) Find the unit tangent vector $\hat{\mathbf{T}}(t)$ in general and at $(0, 1, 0)$.

Answer:

$$\hat{\mathbf{T}}(t) = \frac{\operatorname{sech} t\,\hat{\mathbf{i}} + \tanh t\,\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{2}} \quad \text{and} \quad \hat{\mathbf{T}}(0) = \frac{\hat{\mathbf{i}} - \hat{\mathbf{k}}}{\sqrt{2}}.$$

B) Find the length of the curve $\mathbf{r}(t)$ where $-1 \le t \le 1$.

Answer: $L = 2\sqrt{2}\sinh(1)$.

More Practice Problems

1. Consider the pair of parametric curves

$$\mathbf{r}_1(t) = (t, 1 - t, 3 + t^2) \quad \text{and} \quad \mathbf{r}_2(t) = (3 - t, t - 2, t^2).$$

A) Show that the two curves intersect. At what point in $\mathbb{R}^3$ does this occur?

Answer: At (1,0,4).

B) For $\mathbf{r}_1(t)$, find the equation of the tangent line at the point of intersection in part (A).

Answer: $x = 1 + t, \quad y = -t, \quad \text{and} \quad z = 4 + 2t.$

2. Consider the helix $\mathcal{C}$

$$\mathbf{r}(t) = 3\sin t\,\hat{\mathbf{i}} + 3\cos t\,\hat{\mathbf{j}} + 4t\,\hat{\mathbf{k}}$$

where $t \geq 0$.

A) Reparameterize the curve $\mathcal{C}$ in terms of arc length to deduce a formula for $\mathbf{r}(s)$ for $s \geq 0$.

Answer: $\mathbf{r}(s) = 3\,\sin\left(\frac{s}{5}\right)\,\hat{\mathbf{i}} + 3\,\cos\left(\frac{s}{5}\right)\,\hat{\mathbf{j}} + \frac{4s}{5}\,\hat{\mathbf{k}}.$

B) Explain in words the meaning of $\mathbf{r}(s)$ for a given value of s. How does it differ from $\mathbf{r}(t)$ for a given value of t?

3. Find the length of the curve

$$\mathbf{r}(t) = \ln t\,\hat{\mathbf{i}} + \frac{1}{2}t^2\hat{\mathbf{j}} + \sqrt{2}t\,\hat{\mathbf{k}}$$

when t ranges over the interval $[1, 2]$.

Answer: $L = \frac{3}{2} + \ln 2.$

4. Consider the curve

$$\mathbf{r}(t) = t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + (1 + t^2)\,\hat{\mathbf{k}}.$$

Find the curvature $\kappa(t)$ for $\mathbf{r}(t)$ **and** find at which value of t the curvature $\kappa(t)$ is maximized.

Answer: The maximum is $\kappa = 1$ at $t = 0$.