Math 245 - Multivariate Calculus - Examination #1 - Fall 2024 - Solution

1. (20 total points) Please circle either T (true) or F (false) for each of the below statements. Each correct answer is worth 2 points. There is no partial credit or penalty for guessing. You DO NOT need to show any work. However, space is provided for any calculations you need to perform to help you decide on an answer. **Answers are in BOLD.**

I) T **F** Given a vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^3$, $(\mathbf{a} - \mathbf{b}) \perp (\mathbf{a} + \mathbf{b})$.

II) T **F** For vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^3$, $(\mathbf{a} \times \mathbf{b})/\mathbf{a}$ makes sense.

III) **T** F The plane $7x - 4y + 5z = 14$ intersects the x-axis at $(2, 0, 0)$.

IV) **T** F The osculating plane is perpendicular to the normal plane of a curve $\mathbf{r}(t)$.

V) **T** F The line $\mathcal{L}_1 :\ x = -1 + 2t,\ y = 1 - 4t,\ z = 5t$ is parallel to the line $\mathcal{L}_2 :\ x = 1 + 4t,\ y = 5 - 8t,\ z = 38 + 10t$.

VI) T **F** The curve $\mathbf{r}(t) = \ln(1+t)\,\hat{\mathbf{i}} + 2\cos \pi\, t\,\hat{\mathbf{j}} + \left(e^t + 3\right)^{\frac{1}{2}}\ \hat{\mathbf{k}}$ passes through the point $(0, -2, 2)$.

VII) **T** F $x^2 - 2y^2 - z = 0$ is the equation of a hyperbolic paraboloid.

VIII) **T** F The curve

$$\mathbf{r}(t) = 2\cos(4t)\,\hat{\mathbf{i}} + \sqrt{5}\,t\,\hat{\mathbf{j}} + 2\sin(4t)\,\hat{\mathbf{k}}$$

is a helix with the y-axis as the axis of symmetry.

IX) **T** F The projection of $\hat{\mathbf{i}} + \hat{\mathbf{k}}$ onto $\hat{\mathbf{i}} - \hat{\mathbf{k}}$ is $\mathbf{0}$.

X) T **F** $\mathbf{r}(t) = e^{t^2-1}\,\hat{\mathbf{i}} + \sec(\pi t)\,\hat{\mathbf{j}} - \tan\left(\frac{\pi t}{4}\right)\,\hat{\mathbf{k}}$ is not continuous at $t = 0$.

2. (16 total points) Let $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

A) (8 points) Compute $\mathbf{a} - \mathbf{b}$ and $\hat{\mathbf{b}}$.

Solution:

$$\mathbf{a} - \mathbf{b} = (2, 3, 2) - (1, -2, 2) = (2 - 1, 3 + 2, 2 - 2) = (1, 5, 0) = \boxed{\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}}.}$$

Also, since $|\mathbf{b}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{1 + 4 + 4} = 3$,

$$\boxed{\hat{\mathbf{b}} = \frac{\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3}.}$$

B) (8 points) Find $\mathbf{a} \cdot \mathbf{b}$. Is $\mathbf{a} \perp \mathbf{b}$?

Solution:

$$\mathbf{a} \cdot \mathbf{b} = (2, 3, 2) \cdot (1, -2, 2) = 2 \cdot 1 + 3 \cdot (-2) + 2 \cdot 2 = 2 - 6 + 4 = \boxed{0.}$$

It follows that YES, $\boxed{\mathbf{a} \perp \mathbf{b}.}$

3. (16 points) Find the parametric **and** symmetric equations of the line of intersection between planes $2x - 3y - z = 3$ and $2x - y + 2z = 5$.

<u>*Solution:*</u> The line of intersection has a direction parallel to the cross product of the plane normals. Specifically, letting $\mathbf{n}_1 = (2, -3, -1)$ and $\mathbf{n}_2 = (2, -1, 2)$ denote the respective plane normals, the direction $\mathbf{u}$ is given by

$$\begin{aligned} \mathbf{u} &= \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \hat{\mathbf{i}} & -\hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -3 & -1 \\ 2 & -1 & 2 \end{vmatrix}, \\ &= \hat{\mathbf{i}}(2(-3) - (-1)(-1)) - \hat{\mathbf{j}}(2 \cdot 2 - 2(-1)) + \hat{\mathbf{k}}(2(-1) - 2(-3)), \\ &= -7\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 4\hat{\mathbf{k}}. \end{aligned}$$

A point on the line must also be a point in common between the two planes. To find such a point, we set $z = 0$ and solve the plane equations to find a point in common:

$$2x - 3y = 3 \quad \text{and} \quad \underbrace{2x - y = 5}_{y = 2x - 5} \quad \Rightarrow \quad 2x - 3\underbrace{(2x - 5)}_{= y} = -4x + 15 = 3 \quad \Rightarrow \quad x = 3.$$

It follows that $y = 2x - 5 = 2(3) - 5 = 1$ so we can conclude that $(3, 1, 0)$ is a point on the line, and within both planes. The parametric equations for the line $\mathbf{r}(t) = (3, 1, 0) + (-7, -6, 4)t$ are thus

$$\boxed{x = 3 - 7t, \quad y = 1 - 6t, \quad z = 4t.}$$

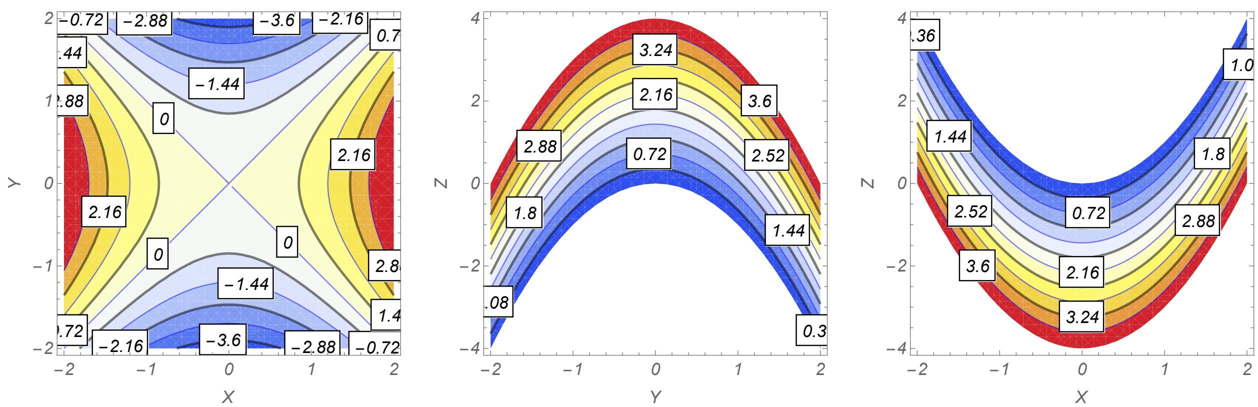
It follows then that the corresponding symmetric equations are

$$\boxed{\frac{x - 3}{-7} = \frac{y - 1}{-6} = \frac{z}{4}.}$$

4. (16 total points) Consider the surface $\mathcal{S}$ described by $x^2 - y^2 - z = 0$.

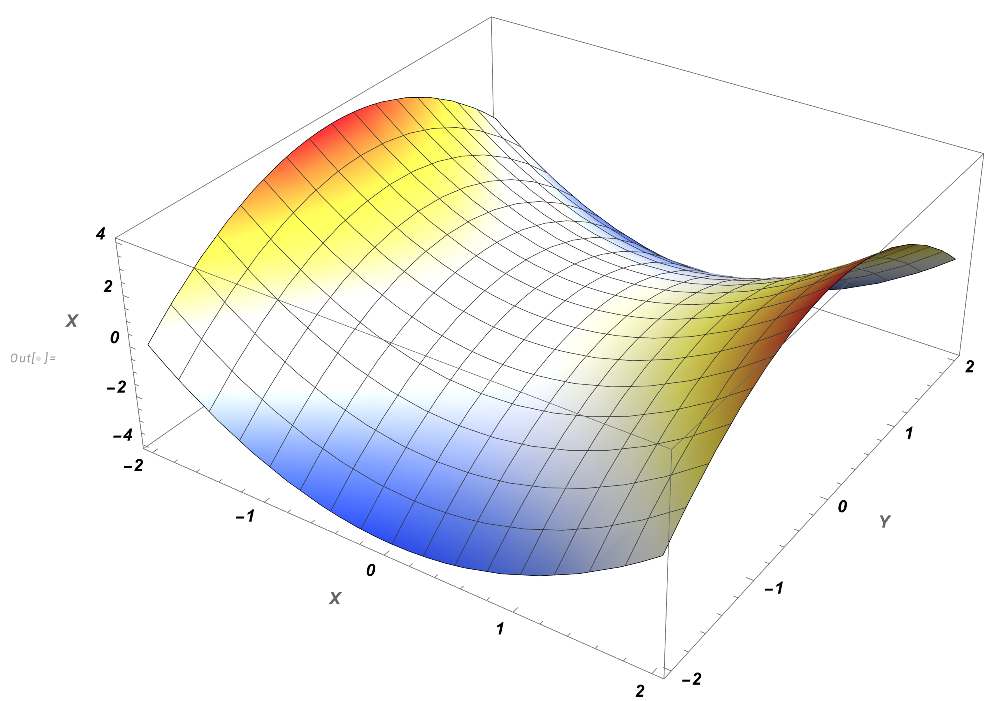
A) (8 points) Find and graph the xy-, xz-, and yz- traces for this surface below. Your answer should have families of graphs below, one for each pair of coordinates.

Solution: Immediately from the equation we see that the xy- and yz- traces are parabolas and the xy-traces are hyperbolas. They are below:



B) (8 points) Use your answer to part (A) to sketch the three-dimensional graph of $\mathcal{S}$ in $\mathbb{R}^3$ and determine its type. What is the name of this surface?

Solution: The surface is an hyperbolic paraboloid.



5. (16 total points) Consider the curve

$$\mathbf{r}(t) = \mathbf{r}(t) = e^{2t}\,\hat{\mathbf{i}} + 2\left(e^{2t}+5\right)\hat{\mathbf{j}} + \left(2e^{2t}-20\right)\hat{\mathbf{k}}.$$

A) (4 points) Find the domain of $\mathbf{r}(t)$. Express your answer using interval notation.

Solution: Since e^{2t} is well-defined for any value of $t \in \mathbb{R}$, it follows that the domain is $\boxed{\mathbb{R} = (-\infty, \infty).}$

B) (10 points) Find the length of the curve over the parameter range $0 \le t \le \ln 2$.

Solution: First note that

$$\mathbf{r}'(t) = 2e^{2t}\,\hat{\mathbf{i}} + 4e^{2t}\,\hat{\mathbf{j}} + 4e^{2t}\,\hat{\mathbf{k}}$$

so that

$$|\mathbf{r}'(t)| = \sqrt{(2e^{2t})^2 + (4e^{2t})^2 + (4e^{2t})^2} = \sqrt{36e^{4t}} = 6e^{2t}.$$

Therefore, the length of the curve as t ranges from $t = 0$ to $t = \ln 2$ is

$$L = \int_0^{\ln 2} \left|\mathbf{r}'(t)\right| \, dt = \int_0^{\ln 2} 6e^{2t}\, dt = 3e^{2t}\Big|_{t=0}^{t=\ln 2} = 3e^{2(\ln 2)} - 3e^{2\cdot 0} = 12 - 3 = \boxed{9.}$$

C) (2 points) Use your answer to Part B) to reparametrize $\mathbf{r}(t)$ in terms of arc length s.

Solution: First observe that $s(t)$ is given by

$$s(t) = \int_0^t \left|\mathbf{r}'(u)\right| \, du = \int_0^t 6e^{2u}\, du = 3e^{2u}\Big|_{u=0}^{u=t} = 3e^{2t} - 3e^{2\cdot 0} = 3e^{2t} - 1.$$

It follows that

$$s = 3e^{2t} - 3 \quad \Rightarrow \quad e^{2t} = \frac{s+3}{3}.$$

Substituting in for e^{2t} in the original expression for $\mathbf{r}(t)$ yields

$$\begin{aligned} \mathbf{r}(s) &= \mathbf{r}(t)\big|_{e^{2t}=(s+3)/3} = \left(\frac{s+3}{3}\right)\hat{\mathbf{i}} + 2\left(\frac{s+3}{3}+5\right)\hat{\mathbf{j}} + \left(2\cdot\frac{s+3}{3} - 20\right)\hat{\mathbf{k}}, \\ &= \boxed{(1, 12, -18) + \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) s.} \end{aligned}$$

6. (16 total points) Consider the curve

$$\mathbf{r}(t) = 4\cos t\,\hat{\mathbf{i}} + \sin\, t\,\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}}.$$

A) (6 points) Find the equation of the plane that is normal to the curve at $(4, 0, 2)$.

<u>*Solution:*</u> First note that $\mathbf{r}(t) = (4, 0, 2)$ implies that $\sin t = 0$ or $t = 0$. Since the normal vector to the normal plane is the tangent vector the curve, it follows that the normal to the normal plane when $t = 0$ is $\mathbf{r}'(0)$. For our case, this becomes

$$\mathbf{r}'(t) = -4\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} - 2\sin t\,\hat{\mathbf{k}} \quad \Rightarrow \quad \mathbf{r}'(0) = (0, 1, 0).$$

Since $(4, 0, 2)$ is necessarily a point on the plane, and $\hat{\mathbf{j}}$ is the plane normal, it follows that the normal plane is

$$\boxed{y = 0.}$$

B) (8 points) Compute $\kappa(t)$, the curvature for any value of t.

<u>*Solution:*</u> Since $\mathbf{r}''(t) = (-4\cos t,\ -\sin t,\ -2\cos\, t)$, it follows that

$$\begin{aligned}
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \hat{\mathbf{i}} & -\hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -4\sin t & \cos t & -2\sin t \\ -4\cos t & -\sin t & -2\cos t \end{vmatrix}, \\
&= \hat{\mathbf{i}}\left(-2\cos^2 t - 2\sin^2 t\right) - \hat{\mathbf{j}}\left(8\sin t\cos t - 8\cos t\sin t\right) + \hat{\mathbf{k}}\left(4\sin^2 t + 4\cos^2 t\right), \\
&= -2\hat{\mathbf{i}} + 4\hat{\mathbf{k}} \quad \Rightarrow \quad |\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{(-2)^2 + 4^2} = 2\sqrt{5}.
\end{aligned}$$

The curvature for any t is thus

$$\begin{aligned}
\kappa(t) &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{5}}{[(-4\sin t)^2 + \cos^2 t + (-2\sin t)^2]^{\frac{3}{2}}}, \\
&= \frac{2\sqrt{5}}{\left[16\sin^2 t + \cos^2 t + 4\sin^2 t\right]^{\frac{3}{2}}} = \frac{2\sqrt{5}}{\left[1 + 19\sin^2 t\right]^{\frac{3}{2}}}.
\end{aligned}$$

C) (2 points) At which point in space on the curve is the curvature κ a maximum?

<u>*Solution:*</u> Since the numerator of $\kappa(t)$ is constant and the denominator is minimized when $\sin^2 t = 0$, it follows that $t = n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$. In particular, the point where curvature is a maximum is when $t = 0$ or, equivalently, at

$$\boxed{\mathbf{r}(0) = (4, 0, 2).}$$